High dimensional risk aggregation: a hierarchical approach with copulas

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Risk aggregation: why?

Swiss Solvency Test (Solvency II is analogous):
one part of the solvency capital requirement (SCR) is

$$ES_{99\%}\left[\frac{Assets(1) - Liabilities(1)}{1+r} - (Assets(0) - Liabilities(0))\right],$$

where $Assets(t) - Liabilities(t)$ is the market consistent valuation of the available capital (= all assets minus all liabilities) at time t.

$Assets(1) - Liabilities(1)$ is random at time 0!

Risk aggregation: why?

One component for solvency capital requirements and risk management:

Calculate the distribution of *Liabilities*(1), which is the sum of all liabilities X_i:

$$Liabilities(1) = \sum_{i=1}^{d} X_i.$$

- X_i: value of liability i at time 1 (random at time 0)
- d: number of liabilities (usually huge!)

Risk aggregation: dependence matters

- Many risks are correlated
- Risks which are uncorrelated in “normal times” become dependent in the extremes.

  Examples:
  - 9/11 terrorist attacks
  - 2011 Tōhoku earthquake (Tsunami, Fukushima, etc.)

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Dependence cannot be ingored!

Popular risk aggregation methodologies

- Variance - Covariance approaches
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- Risk factor models
  - Explicitly modelling risk factors can be difficult
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- Copula models
  - Can theoretically capture all aspects of dependence
  - Finding an adequate copula model is difficult - more details later

Copulas: definition

The joint cumulative distribution function (cdf) of $(X_1, \ldots, X_d)$ can be written as:

$$\mathbb{P}\left[X_1 \leq x_1, \ldots, X_d \leq x_d\right] = C\left(F_1(x_1), \ldots, F_d(x_d)\right),$$

where

- copula function $C : [0,1]^d \to [0,1]$
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- C captures **all** aspects of dependence
- The F_i capture all aspects of the marginal distributions

Copula models

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There are many models for copulas:

- parametric
  - elliptic (Gaussian, t, ...)
  - Archimedean (Clayton, Gumbel, Frank, ...)
  - Vines
  - etc
- nonparametric
  - Bernstein copulas
  - Box copulas
  - Fourier copulas
  - etc

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Hierarchical risk aggregation circumvents these problems.

Hierarchical aggregation: example

We are interested in the aggregate

$$S = X_1 + X_2 + Y$$

Suppose that modelling the copula of (X_1, X_2, Y) is not feasible.

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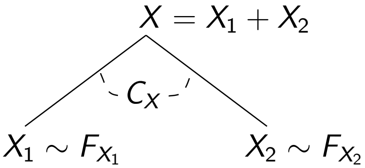

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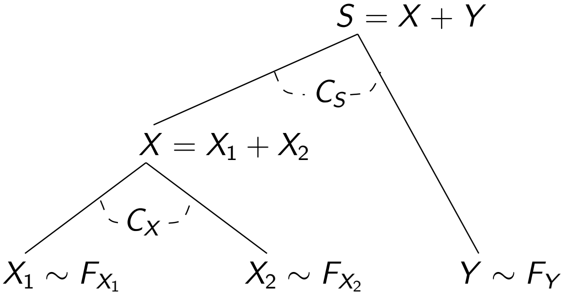

- F_{X_1}, F_{X_2}, F_Y: marginal cdf's of X_1, X_2, Y
- C_X: Copula of (X_1, X_2), C_S: Copula of (X, Y)

Existence & Uniqueness

Theorem: Suppose

- $X_1 \sim F_{X_1}, \quad X_2 \sim F_{X_2}, \quad Y \sim F_Y$ (margins)
- $(X_1, X_2) \sim C_X, \quad (X, Y) \sim C_S$ (copulas)
- $X = X_1 + X_2, \quad S = X + Y$ (aggregation steps)

Then

1. Distribution of (X_1, X_2, Y) exists, but is not unique
2. Distribution of (X_1, X_2) is unique. The same holds for (X, Y) and S

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Consequences:

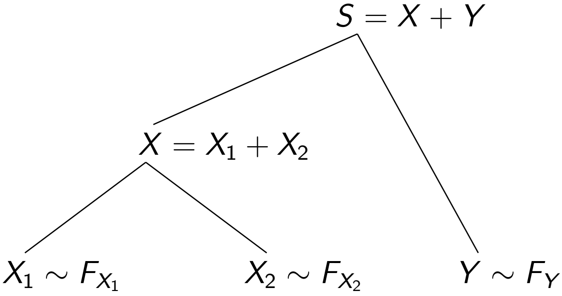
- If we are only interested in the aggregate S: no problem
- Classical capital allocation principles (e.g. Euler) do not work

Existence & Uniqueness

Theorem: Assume additionally that

- $(X_1, X_2) \perp Y | X$: Conditionally on X, (X_1, X_2) is independent of Y.

Then the distribution of (X_1, X_2, Y) is unique.



Modelling point of view

Classical approach: determine one trivariate copula describing the dependence structure of

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Hierarchical approach: determine 2 copulas C_X and C_S

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- These copulas are of lower dimension - "Divide & Conquer"

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Illustration based on a simple problem:

$$(X, Y) \sim F_{X,Y} = C(F_X, F_Y).$$

1. Fix $n \in \mathbb{N}$
2. Simulate independently, for $i = 1, \ldots, n$:
   - $X_i \sim F_X$,
   - $Y_i \sim F_Y$,
   - $U_i = (U_i^1, U_i^2) \sim C$

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   - $X_i \sim F_X$,
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3. Define an approximation of $F_{X,Y}$ by

$$F^n_{X,Y}(x, y) = C^n(F^n_X(x), F^n_Y(y))$$

The atoms of $F^n_{X,Y}$ are given by the samples of X and Y, reordered according to the observed joint ranks in the copula sample.

Reordering algorithm: Sampling margins and Copula

Let $n = 4$.

Sample i.i.d. $X_i \sim F_X$, $i = 1, 2, 3, 4$.

$X_i \sim F_X$		
sample		
3.1		
6.3		
1.4		
5.9		

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sample	rank			
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6.3	4			
1.4	1			
5.9	3			

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$X_i \sim F_X$		$Y_i \sim F_Y$		
sample	rank	sample		
3.1	2	67.9		
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6.3	4	22.8	2	
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Samples of (X, Y): → Samples of $X + Y$:

$(3.1, 43.7) \longrightarrow 3.1 + 43.7 = 46.8$

$(5.9, 67.9) \longrightarrow 5.9 + 67.9 = 73.8$

$(1.4, 12.2) \longrightarrow 1.4 + 12.2 = 13.6$

$(6.3, 22.8) \longrightarrow 6.3 + 22.8 = 29.1$

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$$(5.9, 67.9)$$
$$(1.4, 12.2)$$
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Samples of $X + Y$:
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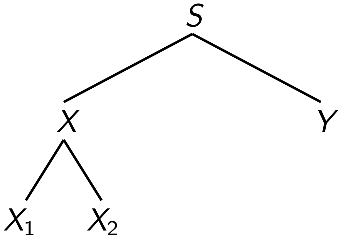
Sampling the aggregation tree



Dependence is described through 2 copulas:

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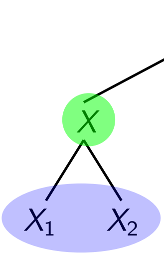

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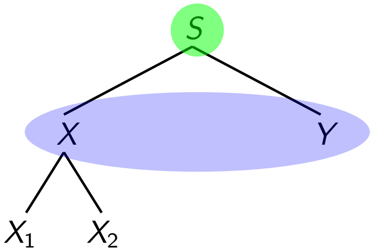
Sampling the aggregation tree



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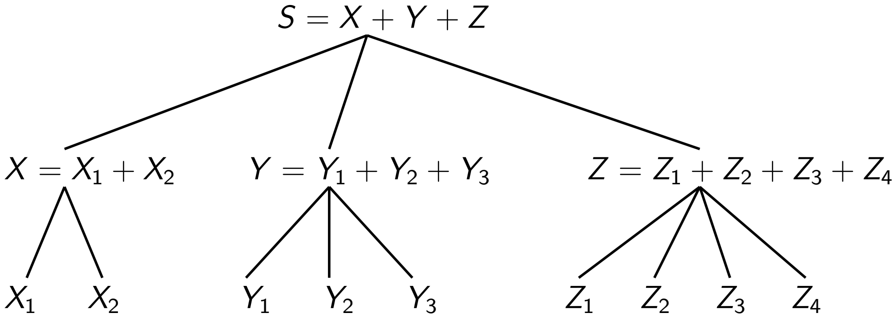
Algorithm:

1. Reorder sample of X_1 and X_2 according to a sample of C_X.
   Sum components to get a sample of X.
2. Reorder sample of X and Y according to a sample of C_S.
   Sum components to get a sample of S.

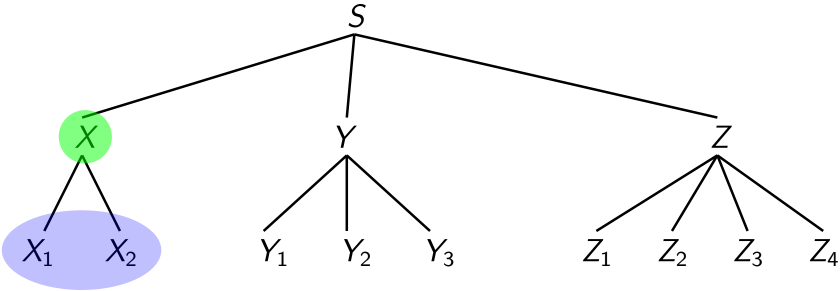
High dimensional example



Dependence is described through 4 copulas:

$$(X_1, X_2) \sim C_X\left(F_{X_1}, F_{X_2}\right)$$
$$(Y_1, Y_2, Y_3) \sim C_Y\left(F_{Y_1}, F_{Y_2}, F_{Y_3}\right)$$
$$(Z_1, Z_2, Z_3, Z_4) \sim C_Z\left(F_{Z_1}, F_{Z_2}, F_{Z_3}, F_{Z_4}\right)$$
$$(X, Y, Z) \sim C_S\left(F_X, F_Y, F_Z\right)$$

Aggregation example, sampling through reordering



Reorder samples of X_1 and X_2 according to the copula C_X.
Calculate samples of X

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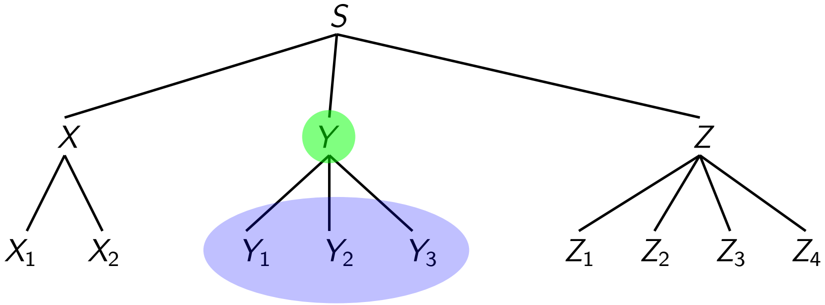

Reorder samples of Y_1, Y_2 and Y_3 according to the copula C_Y.
Calculate samples of Y

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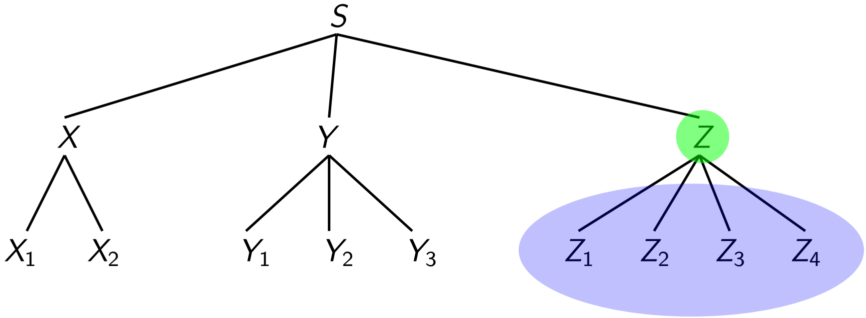
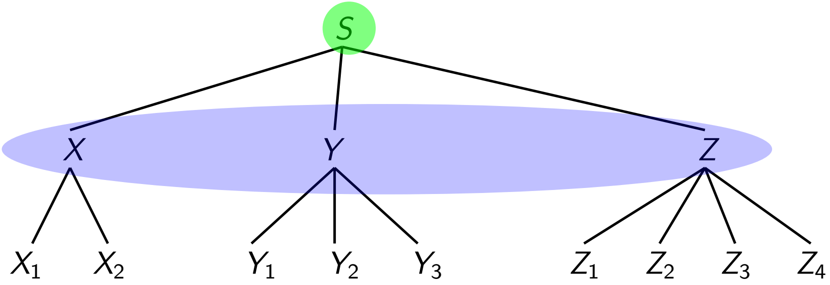


Reorder samples of Z_1, Z_2, Z_3 and Z_4 according to the copula C_Z.
Calculate samples of Z

Aggregation example, sampling through reordering



Through the previous reorderings, we have samples of X, Y and Z!
Reorder those according to the copula C_T, in order to get samples of T.

Aggregation example, convergence

Theorem:
Suppose C_X and C_S are absolutely continuous with *bounded* densities. Then, the empirical cdf F_S^n of S converges to the true distribution F_S:

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|F_S^n(x)-F_S(x)\right| = 0 \qquad \mathbb{P}\text{-a.s.},$$

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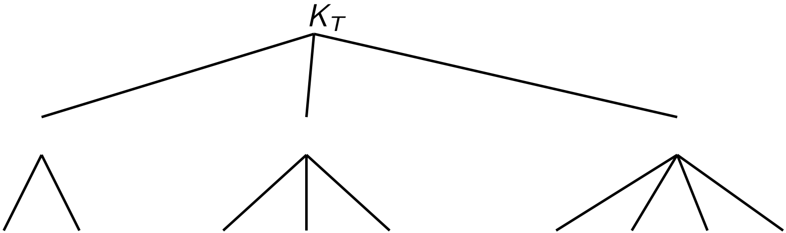
- Why only bounded densities? Underlying set classes **do not** satisfy Vapnik-Červonenkis (VC) property!
- For unbounded densities:
  - works for few examples (e.g. bivariate Clayton)
  - in general: open problem

Capital allocation

Capital allocation is easy: allocate hierarchically

1. Risk capital: K_T



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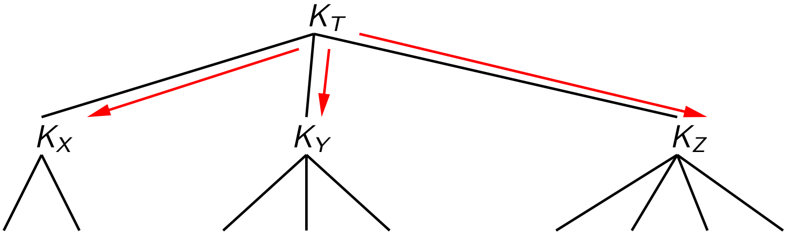

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2. One has a sample of (X, Y, Z)!
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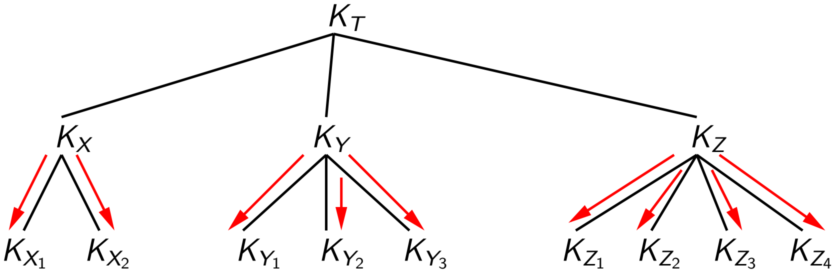
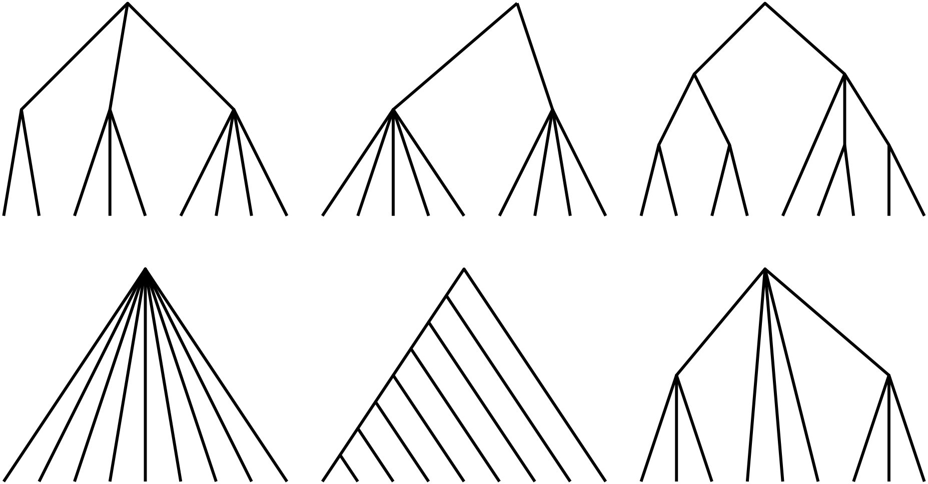
3. One has a sample of (X_1, X_2)!
   Allocate to X_1 and X_2 by splitting K_X. Analogous for Y and Z



How to set the aggregation tree?

For 9 risks, there are 12'818'912 aggregation trees!



Estimation of the tree

Estimating the tree from data: not feasible. Model identification problems!

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Aggregate by risk types. Groupings are inherent due to

- line of business
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Dependence between risks gets weaker the farther they are apart

- Keep the number of aggregation levels low
- Strongest dependencies at the bottom
- Subaggregates with similar roles should be on the same level in the tree.

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As aggregation steps are low dimensional, the number of parameters is much lower than in a full model.

To calibrate copulas, use:

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  - If the tree is cleverly chosen, aggregated variables for each aggregation step have intuitive meaning.
- Regulatory guidelines

Or a combination of these sources of information.

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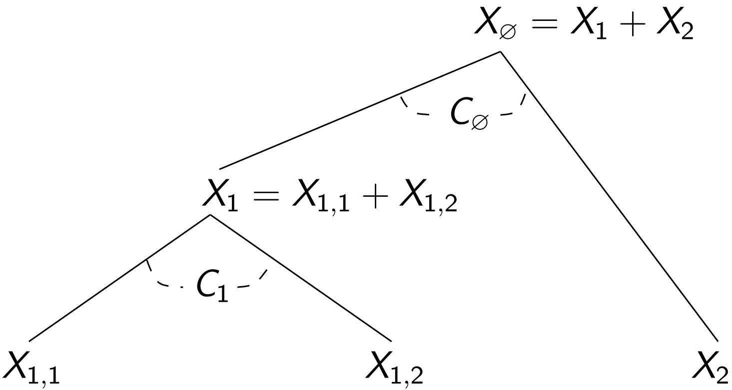
- Estimating the tree: heuristics
- Calibration is easy: only low dimensional problems
- Capital allocation is possible
- The model is graphically and mathematically intuitive: can be explained and defended in front of stakeholders

Thank you.

Appendix: Sampling the whole tree

Up to now: sampling described only for each aggregation step.
How to sample from the whole tree?
Idea: pull back permutations from top to bottom of the tree!



marginal samples: $\overset{X_{1,1}}{\begin{pmatrix}0\\0.1\\0.2\end{pmatrix}}$ $\overset{X_{1,2}}{\begin{pmatrix}0\\1\\2\end{pmatrix}}$ $\overset{X_2}{\begin{pmatrix}0\\10\\20\end{pmatrix}}$

reordering of $X_{1,1}$ and $X_{1,2}$: $\begin{matrix}X_{1,1} & X_{1,2}\end{matrix}$ $\begin{pmatrix}0.1 & 0\\0 & 2\\0.2 & 1\end{pmatrix} \to \sum \to \overset{X_1}{\begin{pmatrix}0.1\\2\\1.2\end{pmatrix}}$

reordering of X_1 and X_2: $\begin{matrix}X_1 & X_2\end{matrix}$ $\begin{pmatrix}2 & 10\\1.2 & 20\\0.1 & 0\end{pmatrix} \to \sum \to \overset{X_\varnothing}{\begin{pmatrix}12\\21.2\\0.1\end{pmatrix}}$

Apply to $X_{1,1}$ and $X_{1,2}$ the permutations which were applied to X_1. I.e., pull back permutations to leaf nodes to construct a joint sample:

$$\begin{matrix}X_{1,1} & X_{1,2} & X_1 & X_2 & X_\varnothing\end{matrix}$$
$$\begin{pmatrix}0 & 2 & 2 & 10 & 12\\0.2 & 1 & 1.2 & 20 & 21.2\\0.1 & 0 & 0.1 & 0 & 0.1\end{pmatrix}$$