Model independent bounds for options pricing
A stochastic control approach

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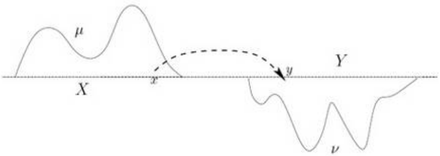
Outline

Problème des déblais et des remblais

Gaspard Monge (1746-1818) formulated in 1781 the following engineering problem :

Transportation of mass for the lowest cost,
given initial and final distribution



Analytic formulation

- Initial distribution : probability measure μ_0
- Final distribution : probability measure μ_1

Problem :

$$\min_{T \in \mathcal{T}(\mu_0,\mu_1)} \int c(x, T(x))\mu_0(dx)$$

where $\mathcal{T}(\mu_0, \mu_1)$ of all maps $T : x \longmapsto T(x)$ such that

$$\mu_1 \quad = \quad \mu_0 \circ T^{-1}$$

Probabiblistic formulation

• Leonid Vitaliyevich Kantorovich (1912-1986) provided in 1942 a formulation which does not involve the transportation scheme T :

$$\inf_{\pi \in \Pi(\mu_0,\mu_1)} \int c(x,y)\pi(dx,dy)$$

where $\Pi(\mu_0,\mu_1)$ is the collection of all joint probability measures with marginals μ_0 and μ_1

Exemple : $c(x,y) = |x-y|^2 \Longrightarrow$ maximization of correlations :

$$sup_{\pi \in \Pi(\mu_0,\mu_1)} \mathbb{E}^\pi[XY]$$

The discrete version

- The discrete version is called Optimal assignment problem and is widely studied in the operations research literature :

- Assignment persons / jobs

- Hedge funds : assignment of executed orders to different funds...

Kantorovich duality

Duality in linear programming, Legendre-Fenchel duality...

- Penalize the constraint $\Longrightarrow$ Lagrangian
- $\min_\pi \max_\lambda = \max_\lambda \min_\pi$

This leads to the dual formulation :

$$\sup \left\{ \int \psi(y)\mu_1(dy) - \int \varphi(x)\mu_0(dx) : \psi(y) - \varphi(x) \le c(x,y) \right\}$$

Economic interpretation :

- $\varphi(x)$ bid price at x (buy at this price)
- $\psi(y)$ ask price at y (sell at this price)

Outline

Controlled dynamics

- W : Brownian motion with values in $\mathbb{R}^d$

- $\mathcal{U}$: collection of all control processes $\nu = (b, \sigma^2)$ prog. measurable processes (with appropriate dimensions) valued in U (closed convex subset of $\mathbb{R}^k$)

- Controlled dynamics :

$$dX_t = b_t dt + \sigma_t dW_t$$

We shall impose the transportation to follows the above dynamics for some choice of control (b, σ^2)

Mikami and Thieullen considered the case $U = \mathbb{R}^{k'} \times \{I_d\}$ (fixed diffusion coefficient)

Optimal transportation problem

Optimal transportation under controlled stochastic dynamics :

$$V_0 \quad := \quad \inf_{\nu \in \mathcal{U}(\mu_0,\mu_1)} \mathbb{E}\Big[\int_0^1 L(t, \underline{X}_t, \nu_t)dt\Big]$$

where $\underline{X}_t := (X_u)_{u \le t}$ and

$$\mathcal{U}(\mu_0,\mu_1) \quad := \quad \big\{\nu \in \mathcal{U} : \; X_0 \sim \mu_0 \text{ and } X_1 \sim \mu_1\big\}$$

Dual formulation (Xiaolu TAN)

Assume $u \longmapsto L(t, x_., u)$ convex and coercive then

$$V_0 = \sup_{\varphi_1 \in C_b^0} \int \varphi_1(y)\mu_1(dy) + \int \varphi_0(x)\mu_0(dx)$$

where $\varphi_0(x) = \varphi(0, x)$ and φ defined by

$$\varphi(t, x) := \inf_{\nu \in \mathcal{U}} \mathbb{E}\Big[\int_t^1 L(s, \underline{X}_s^{t,x}, \nu_s) ds - \varphi_1(X_1^{t,x}) \Big]$$

Unconstrained stochastic control problem

The Markov case : control of PDEs

In the Markov case

$$L(s, \underline{x}_s, u) = L(s, x_s, u)$$

The value function φ can be characterized by the dynamic programming equation :

$$\partial_t \varphi + \inf_{(b,\sigma^2)\in U} \left\{ b \cdot D\varphi + \frac{1}{2}\mathrm{Tr}\left[\sigma^2 D^2 \varphi\right] + L(t, x, b, \sigma^2) \right\} = 0$$

$$\varphi(1, x) = \varphi_1(x)$$

$\Longrightarrow$ Numerical schemes...

Outline

Unspecified volatility process : abstract formulation

- $\Omega = \{\omega \in C(\mathbb{R}_+) : \omega(0) = 0\}$,
- B coordinate process, $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$, $\mathbb{P}_0$: Wiener measure

Suppose that B is only known to be a continuous local martingale with quadratic variation $\langle B \rangle$ a.c. wrt Lebesgue. Let

$$\mathcal{P} := \left\{ \mathbb{P}_0 \circ \left(\int_0^\cdot \sigma_t dB_t\right)^{-1} : \ \sigma \ \mathbb{F}-\text{prog. meas.}, \ \int_0^T |\sigma_t|^2 dt < \infty \right\}$$

- Zero interest rate, and risky asset defined by :

$$dS_t \ = \ S_t dB_t, \ \ \mathcal{P} - \text{q.s.}$$

where $\mathcal{P}-$quasi-surely means $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}$

Model-free bounds

- Super-hedging and Sub-hedging problems of $\mathcal{F}_T-$meas. r.v. ξ

$$U := \inf\left\{X_0 : \ \exists\, H \in \mathcal{H} : X_0 + \int_0^T H_t dB_t \geq \xi, \ \mathcal{P} - \text{q.s.}\right\}$$

$$L := \sup\left\{X_0 : \ \exists\, H \in \mathcal{H} : X_0 + \int_0^T H_t dB_t \leq \xi, \ \mathcal{P} - \text{q.s.}\right\}$$

- the portfolio $H \in \mathcal{H}$ does not depend on a particular $\mathbb{P} \in \mathcal{P}$...

- Denis-Martini 2005 and Peng 2007 for the bounded volatility case $(\underline{\sigma} \leq \sigma_. \leq \overline{\sigma})$

More appealing formulation

- Control process is the volatlity σ valued in $\mathbb{R}_+$
- Controlled process is the underlying asset price process :

$$dS_t = S_t \sigma_t dW_t$$

- Model-free bounds for the derivative $\xi(\underline{S}_T)$

$$U := \inf \left\{ X_0 : \exists H \in \mathcal{H} : X_0 + \int_0^T H_t dB_t \geq \xi, \text{ a.s. for all } \sigma_. \right\}$$

$$L := \sup \left\{ X_0 : \exists H \in \mathcal{H} : X_0 + \int_0^T H_t dB_t \leq \xi, \text{ a.s. for all } \sigma_. \right\}$$

Dual formulation

Consider the superhedging problem

$$U_0 := \inf\left\{X_0 : \; X_0 + \int_0^T H_t dB_t \geq \xi, \; \mathcal{P}-\text{q.s. for some } H \in \mathcal{H}_0\right\}$$

where

$$\mathcal{H}_0 := \left\{H : \; H \in \mathbb{H}^2_{loc}(\mathbb{P}) \text{ and } X^H \geq \text{Mart}^{\mathbb{P}}, \; \forall \mathbb{P} \in \mathcal{P}\right\}$$

Theorem (Soner, T., Zhang 2010) For all $\xi \in \mathcal{L}^\infty_{\mathcal{P}}$:

$$U_0 = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi]$$

and existence holds for the problem U_0

Bounds with no further information

For $\xi = g(S_T)$, we find

$$U_0(\xi) = g^{\text{conc}}(S_0) \quad \text{and} \quad L_0(\xi) = g^{\text{conv}}(S_0)$$

and the corresponding hedging strategy H^* is of type Buy-and-Hold

$\Longrightarrow$ dynamic hedge does not help to reduce the superhedging cost...

Model-free bounds with more information

• Suppose that prices of $T-$maturity call options for all possible strikes $c(k), k \geq 0$ are observed and tradable. Then the map

$$k \longmapsto c(k) := \mathbb{E}[(S_T - k)^+]$$

characterizes the distribution $S_T \sim_{\mathbb{P}} \mu$ by $\mu([k,\infty)) = -c'(k)$

• The no-arbitrage bounds can be improved to

$$U(\mu) := \inf\left\{X_0 : \exists\, H \in \mathcal{H},\ \lambda \in \Lambda : X_T^{H,\lambda} \geq \xi,\ \mathcal{P}-\text{q.s.}\right\}$$

$$L(\mu) := \sup\left\{X_0 : \exists\, H \in \mathcal{H},\ \lambda \in \Lambda : X_T^{H,\lambda} \leq \xi,\ \mathcal{P}-\text{q.s.}\right\}$$

where $\Lambda = \{\text{bdd measurable functions}\}$ and

$$X_T^{H,\lambda} := X_0 + \int_0^T H_t dB_t + \lambda(S_T) - \mu(\lambda)$$

$$\text{“} \; = X_0 + \int_0^T H_t dB_t + \int \lambda''(k)[(S_T - k)^+ - c(k)]dk \text{ ”}$$

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where $\Lambda = \{\text{bdd measurable functions}\}$ and

$$\begin{aligned} X_T^{H,\lambda} &:= X_0 + \int_0^T H_t dB_t + \lambda(S_T) - \mu(\lambda) \\ \text{“} &= X_0 + \int_0^T H_t dB_t + \int \lambda''(k)[(S_T - k)^+ - c(k)]dk\text{”} \end{aligned}$$

Duality and stochastic control

- Notice that

$$U(\mu) = \inf_{\lambda\in\Lambda} \inf\left\{X_0 : \exists\, H \in \mathcal{H}, X_T^H \geq \xi - \lambda(S_T) + \mu(\lambda),\ \mathcal{P}-\text{q.s.}\right\}$$

$$L(\mu) = \sup_{\lambda\in\Lambda} \sup\left\{X_0 : \exists\, H \in \mathcal{H}, X_T^H \leq \xi - \lambda(S_T) + \mu(\lambda),\ \mathcal{P}-\text{q.s.}\right\}$$

- Then, the previous duality implies that

$$\begin{aligned} U(\mu) &:= \inf_{\lambda\in\Lambda} \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\xi - \lambda(S_T) + \mu(\lambda)\Big] \\ L(\mu) &:= \sup_{\lambda\in\Lambda} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\xi - \lambda(S_T) + \mu(\lambda)\Big] \end{aligned}$$

- For every fixed λ : standard stochastic control problem...

The Skorohod Embedding Problem

Previous literature by D. Hobson, L.C.G. Rogers, A. Cox, J. Obloj, B. Dupire, P. Carr, R. Lee

- adressed this problem by using results from the SEP :

Given μ probability measure on $\mathbb{R}$ with $\int |x|\mu(dx) < \infty$
Find a stopping time τ such that
$B_\tau \sim \mu$ and $\{B_{t\wedge\tau}, t \geq 0\}$ UI martingale

(Hall, Monroe, Azéma, Yor, Perkins, Chacon, Walsh, Rost, Root, Bass, Vallois)

- More than twenty known solutions (see Obloj for a survey)

Example : the Azéma-Yor solution

Define the barycenter function :

$$b(x) \quad := \quad \frac{\int_x^\infty s\mu(ds)}{\int_x^\infty \mu(ds)}$$

Then, the Azéma-Yor solution of the SEP is :

$$\tau_{\text{AY}} := \inf\left\{t > 0 : \; B_t^* > b(B_t)\right\}, \quad \text{with} \quad B_t^* := \max_{s \le t} B_s$$

i.e. $\{B_{t\wedge\tau_{\text{AY}}}, t \ge 0\}$ is UI martingale and $B_{\tau_{\text{AY}}} \sim \mu$.

For later use

$$\frac{c(\zeta)}{x-\zeta} \quad \text{decreases on} \quad \left[0, b^{-1}(x)\right] \quad \text{and increases on} \quad \left[b^{-1}(x), x\right)$$

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Connection with our problem

- $(M_t)_{t\geq 0}$ continuous martingale, $M_0 = 0$ and $M_T \sim \mu$,

Then $M_t = B_{\langle M\rangle_t}$ and $\langle M\rangle_T$ solution of SEP

- Let τ be a solution of SEP

Then $M_t := B_{\frac{t}{T-t}\wedge\tau}$ is a continuous martingale, $M_0 = 0$ and $M_\tau \sim \mu$.

Optimality of the Azéma-Yor solution

Let g be C^1 nondecreasing, and define :

$$H(m,x) \quad := \quad \int_0^m g'(r)\frac{r-x}{r-b^{-1}(r)}dr \quad \text{so that}$$

- $\{H(B_t^*, B_t), t \geq 0\}$ is a local martingale
- $g(m) - H(m,x) \leq g\big(b(x)\big) - H\big(b(x),x\big) =: G(x)$

Then

$$g\big(B_\tau^*\big) \quad \leq \quad \underbrace{H\big(B_\tau^*, B_\tau\big)}_{(\text{loc. mart.})_\tau} + G\big(B_\tau\big)$$

and

$$\sup_{\mathbb{P}\in\mathcal{P}(\mu)} \mathbb{E}^{\mathbb{P}}\left[g(S_T^*)\right] = \max_{\tau\in\mathcal{T}(\mu)} \mathbb{E}\left[g\big(B_\tau^*\big)\right] = \int G(x)\mu(dx) = \mathbb{E}\left[g\big(B_{\tau_{AY}}^*\big)\right]$$

For general derivatives : Numerical approximation

• For every fixed λ, build a numerical scheme to approximate the value function

$$u^\lambda \quad := \quad \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}\left[\xi - \lambda(S_T)\right]$$

this is a singular stochastic control, which can be characterized by an elliptic equation... finite differences

• Minimize over λ :

$$\inf_{\lambda} \mu(\lambda) - u^\lambda$$

numerical approximation by the gradient projection algorithm...

Xiaolu TAN...

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Lookback options with one known marginal distribution

Let :

$$\xi \;=\; g(S_T, S_T^*) \quad \text{where} \quad S_T^* := \max_{t \le T} S_t$$

Our main results :

recover the known explicit bounds in this context (so far, those induced by the Azéma-Yor embedding)

Hobson 98, Hobson and Kimmel 2011

Dynamic Programming Equation – Fixed λ

Given $\lambda(.)$, the stochastic control problem is

$$u^\lambda(t,s,m) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^P\left[g(S_T^{t,s}, M_T^{t,s,m}) - \lambda(S_T^{t,s})\right], \quad M_T^{t,s,m} := m \vee \max_{[t,T]} S^t$$

$\Longrightarrow$ Optimal stopping representation and DPE characterization :

$$u^\lambda(t,s,m) = u^\lambda(s,m) = \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[g(S_\tau^s, M_\tau^{s,m}) - \lambda(S_\tau^s)\right]$$

$$\min\left\{u^\lambda - (g-\lambda), -u^\lambda_{ss}\right\} = 0 \text{ for } 0 < s < m$$

$$u^\lambda_m(m,m) = 0$$

Optimal stopping policy τ^* (if exists) $\Longrightarrow$ Optimal volatility process σ^*

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Lagrange Multipliers reduction

We first prove that :

$$U(\mu) = \inf_{g_{ss}-\lambda''\leq 0} \sup_{\tau\in\mathcal{T}} \mu(\lambda) + \mathbb{E}\left[g(M_\tau^{s,m}) - \lambda(S_\tau^s)\right]$$

Solving the optimal stopping problem

Given λ with $g - \lambda$ concave :

$$u^\lambda(s,m) \quad = \quad \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[g(S^s_\tau, M^{s,m}_\tau) - \lambda(S^s_\tau)\right]$$

assume g is C^1 increasing, then we expect that there is a boundary ψ (continuous increasing) so that :

$$u^\lambda(s,m) = (g-\lambda)(s \wedge \psi(m), m) + (g-\lambda)_s(s \wedge \psi(m), m)(s - s \wedge \psi(m))$$

The Neuman condition provides the (ODE) for ψ :

$$(g-\lambda)_{ss}(\psi(m), m)\psi'(m) = \frac{g_m(\psi(m), m)}{m - \psi(m)} + g_{sm}(\psi(m), m)$$

The explicit upper bound

W.l.o.g. $\lambda(S_0) = 0$, then :

$$
\begin{aligned}
U(\mu) &= g(S_0) + \inf_{\lambda''\geq 0} \mu(\lambda) + \int_{\psi(M_0)}^{S_0\vee\psi(M_0)} (s-k)\lambda''(k)dk \\
&\geq g(S_0) + \inf_{\lambda''\geq 0} \int c(k)\lambda''(k)dk \\
&= g(S_0) + \inf_{\lambda''\geq 0} \int c(\psi(x))\lambda''(\psi(x))\psi'(x)dx \\
&= g(S_0) + \inf_{\psi\in\ldots} \int c(\psi(x))\frac{g'(x)}{x-\psi(x)}dx \quad (ODE) \\
&\geq g(S_0) + \int \inf_{\psi<x} \frac{c(\psi)}{x-\psi} g'(x)dx \longrightarrow \psi^*(x) = b^{-1}(x)
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A recursive sequence of control problems

Given n functions $\lambda = (\lambda_i)_{1\le i\le n}$ define :

$$u^\lambda := u^0(S_0, M_0) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[g(M_{t_n}) - \sum_{i=1}^n \lambda_i(S_{t_n})\right]$$

Optimal upper bound given that $S_{t_i} \sim \mu_i$, $i \le n$:

$$\inf_{(\lambda_i)_{1\le i\le n}} \sum_{i=1}^n \mu_i(\lambda_i) + u^\lambda$$

Then, we introduce for $i = 1, \ldots, n$:

$$\begin{aligned} u^n(s,m) &:= g(m) \\ u^{i-1}(s,m) &:= \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[u^i(S_{t_n}, M_{t_n}) - \lambda_i(S_{t_n}) \Big| (S,M)_{t_{n-1}} = (s,m)\right] \end{aligned}$$

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Extension of Peskir's maximality principle

Lemma Optimization can be restricted to those λ_i's such that $\lambda^i - u^i$ is strictly convex for all $i = 1, \ldots, n$

Theorem For λ^i s.t. $\lambda^i - u^i$ is strictly convex, $u^{i-1} < \infty$ iff there is a maximal solution ψ_i of the ODE

$$\psi_i' \left(\lambda_i''(\psi_i) - u^i_{ss}(\psi_i, m)\right) \quad = \quad u^i_{sm}\left(\psi_i, m\right) + \frac{u^i_m\left(\psi_i, m\right)}{m - \psi_i}$$

which stays strictly below the diagonal $\psi_i(m) < m$, $m \geq 0$.
In this case :

$$u^{i-1}(s, m) = u^i(s, m) - \lambda_i(s) + \int_{\psi_i(m)}^{s \vee \psi_i(m)} (s-k)\left(\lambda_i''(k) - u^i_{ss}(k, m)\right) dk$$

Explicit finite dimensional optimization problem

Proceeding as in the case of one marginal, we arrive at the optimization problem :

$$U(\mu_1, \ldots, \mu_n)$$
$$\geq \int \inf_{0<\psi_\cdot<x} \; \frac{c_n(\psi_n)}{x-\psi_n} + \sum_{i=1}^{n-1} \left(\frac{c_i\left(\overline{\psi}_i\right)}{x-\overline{\psi}_i} - \frac{c_i\left(\overline{\psi}_{i+1}\right)}{x-\overline{\psi}_{i+1}} \right) \mathbb{1}_{\{\overline{\psi}_i<\overline{\psi}_{i+1}\}}$$

where

$$\overline{\psi}_i := \psi_i \wedge \ldots \wedge \psi_n \quad \text{for all} \quad i = 1, \ldots, n$$

Optimal upper bound given two marginals

The case $n = 2$ reduces to :

$$\inf_{\psi_i < x} \left\{ c_2(\psi_2) - \mathbb{1}_{\{\psi_2 > \psi_1\}} \left(\frac{c_1(\psi_2)}{x - \psi_2} - \frac{c_1(\psi_1)}{x - \psi_1} \right) \right\}$$

which recovers Hobson and Rogers 1998 $\Longrightarrow$

- $\psi_1(x) = b_1^{-1}(x)$ (Azéma-Yor)
- $\psi_2(x)$ defined by

$$\inf_{\psi_2 < x} \left\{ c_2(\psi_2) - \mathbb{1}_{\{\psi_2 > \psi_1(x)\}} \left(\frac{c_1(\psi_2)}{x - \psi_2} - \frac{c_1(\psi_1(x))}{x - \psi_1(x)} \right) \right\}$$

The $n-$marginals problem reduces to $2-$marginals problems

- First, minimize wrt $\overline{\psi}_1 \leq \overline{\psi}_2$, given $\underline{\psi}_2, \ldots, \underline{\psi}_n < x$:

$$\min_{\overline{\psi}_1 \leq \overline{\psi}_2} \left(\frac{c_1\left(\overline{\psi}_1\right)}{x - \overline{\psi}_1} - \frac{c_1\left(\overline{\psi}_2\right)}{x - \overline{\psi}_2} \right) \mathbb{1}_{\{\overline{\psi}_1 < \overline{\psi}_2\}}$$

- For $i \leq n$, assume $\overline{\psi}^*_{i-1}(x)$ does not depend on $\overline{\psi}_i$ on $\{\overline{\psi}^*_{i-1}(x) < \overline{\psi}_i\}$ for all $x \geq 0$. Then with $\psi_{n+1}(x) = x$:

$$\min_{\overline{\psi}_i \leq \overline{\psi}_{i+1}} \frac{c_i\left(\overline{\psi}_i\right)}{x - \overline{\psi}_i} - \left(\frac{c_{i-1}\left(\overline{\psi}_i\right)}{x - \overline{\psi}_i} - \frac{c_{i-1}\left(\overline{\psi}^*_{i-1}(x)\right)}{x - \overline{\psi}^*_{i-1}(x)} \right) \mathbb{1}_{\{\overline{\psi}^*_{i-1}(x) < \overline{\psi}_i\}}$$

These steps are similar to the $2-$marginals case of Brown, Hobson and Rogers...

The $n-$marginals problem reduces to $2-$marginals problems

- First, minimize wrt $\overline{\psi}_1 \leq \overline{\psi}_2$, given $\underline{\psi}_2, \ldots, \underline{\psi}_n < x$:

$$\min_{\overline{\psi}_1 \leq \overline{\psi}_2} \left(\frac{c_1\left(\overline{\psi}_1\right)}{x - \overline{\psi}_1} - \frac{c_1\left(\overline{\psi}_2\right)}{x - \overline{\psi}_2} \right) \mathbb{1}_{\left\{\overline{\psi}_1 < \overline{\psi}_2\right\}}$$

- For $i \leq n$, assume $\overline{\psi}^*_{i-1}(x)$ does not depend on $\overline{\psi}_i$ on $\{\overline{\psi}^*_{i-1}(x) < \overline{\psi}_i\}$ for all $x \geq 0$. Then with $\psi_{n+1}(x) = x$:

$$\min_{\overline{\psi}_i \leq \overline{\psi}_{i+1}} \frac{c_i\left(\overline{\psi}_i\right)}{x - \overline{\psi}_i} - \left(\frac{c_{i-1}\left(\overline{\psi}_i\right)}{x - \overline{\psi}_i} - \frac{c_{i-1}\left(\overline{\psi}^*_{i-1}(x)\right)}{x - \overline{\psi}^*_{i-1}(x)} \right) \mathbb{1}_{\left\{\overline{\psi}^*_{i-1}(x) < \overline{\psi}_i\right\}}$$

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Solving the $n-$marginals problem

Step 1 : $\psi_1^* = b_1^{-1}$

Step i : minimization over $\overline{\psi}_i \leq \overline{\psi}_{i+1}$, given $\overline{\psi}_{i-1}^*$:

• If $b_i \leq b_{i+1}$, we find $\psi_i^* = b_i^{-1}$ (this recovers Madan and Yor)

• In the general case (not covered in the literature), we rely on :

Lemma Let $i = 2, \ldots, n$ be fixed, assume that $c_i \geq c_{i-1}$, and let $\overline{\psi_i}^*(x)$ be any minimizer of the Step i problem. Then, the function $x \longmapsto \overline{\psi_i}^*(x)$ is nondecreasing

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- Continuous-time limit (extension of Madan-Yor 02)
- Construct martingale processes corresponding to the bound
- Lower/upper bound on Variance calls given 1 (and more generally n) marginals,
- General theory for the treatment of stochastic control problems given marginals, i.e. optimal transportation along controlled stochastic dynamics

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