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On refracted stochastic processes and the analysis of insurance risk

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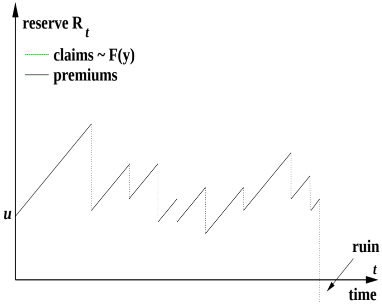
Classical Collective Risk Model



$$R_t = u + c\,t - \sum_{n=1}^{N(t)} X_n$$

$N(t)\ldots$ homogeneous Poisson process (λ)
$X_n\ldots$ iid random variables (d.f. F)
$c\ldots$ premium density

Ruin Probability

$$\boxed{\psi(u) = 1 - \phi(u) = \mathbb{P}\left(\inf_{t\geq 0} R_t < 0 \,\middle|\, R_0 = u\right)}$$

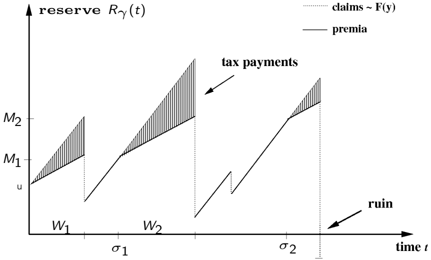
How do tax payments change the ruin probability?

(joint work with Hipp, Borst & Boxma & Resing)

Definition of "profit" of insurance company?

(equalization reserves, claims reserves (IBNR, RBNS,...)) In practice: Tax privileges were reduced recently!

Model: Tax rate $0 \leq \gamma \leq 1$ in "profitable" times:



- Ruin probability $\psi_\gamma(u) = 1 - \phi_\gamma(u)$
- How much taxes does company pay during its lifetime?
- Which taxation rule is most profitable for tax authority?
  Should taxation start when an insurer starts his business?

$\rightarrow$ Simple power relation $\boxed{\phi_\gamma(u) = (\phi_0(u))^{\frac{1}{1-\gamma}}}$

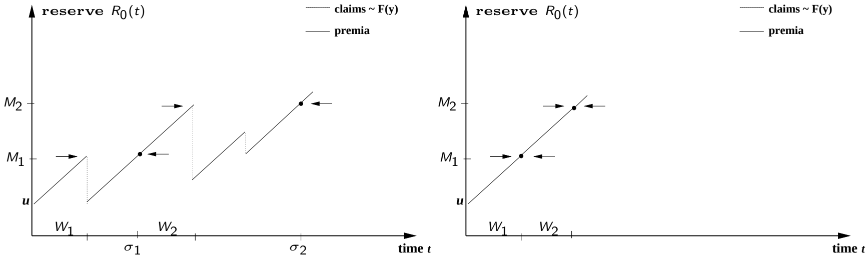
$$\Rightarrow \psi_\gamma(u) \sim \frac{1}{1-\gamma}\, \psi_0(u).$$

Additional solvency capital for tax is asymptotically:

Light tails: $-\frac{1}{R}\log(1-\gamma)$

Pareto(α) tails: $\left((1-\gamma)^{-1/(\alpha-1)} - 1\right)u.$

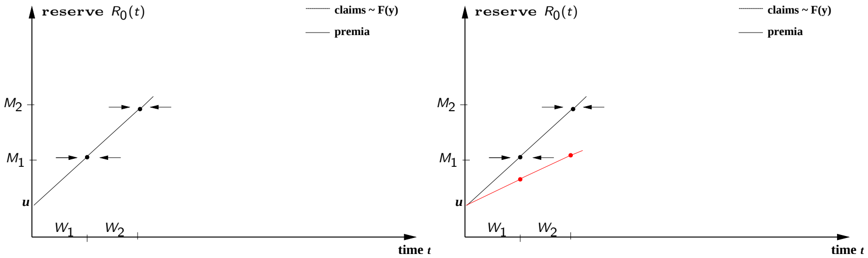
Rescale time: $R_t^* = u + t - \sum_{i=1}^{N_t^*} X_i$ (with N_t^* ...hom. Poisson process (λ/c))



- Link between $\phi_0(u)$ and $V_{\max}$ ...maximum workload during busy period of M/G/1 queue
- Net profit condition $c > \lambda\,\mathbb{E}(X_i) \Leftrightarrow$ traffic intensity $\rho < 1$
- Cut out excursions from running maximum that "survive" $(u \leftrightarrow t)$

Interpretation: $\phi_0(u) = P($no events during "time" interval $[u, \infty)$ of an inhom. Poisson process with time-dependent rate $\alpha(t) = \frac{\lambda}{c}\, P(V_{\max} \geq t)$

$$\phi_0(u) = \exp\Big(-\int_u^\infty \alpha(t)\,dt\Big) = \exp\Big(-\frac{\lambda}{c}\int_u^\infty \mathbb{P}(V_{\max} > t)\,dt\Big)$$



Interpretation: $\phi_\gamma(u) = P$(no events during "time" interval $[u,\infty)$ of an inhom. Poisson process with time-dependent rate $\alpha_\gamma(t) = \frac{\lambda}{c\,(1-\gamma)}\, P(V_{\max} \geq t)$)

$$\phi_\gamma(u) = \exp\left(-\int_u^\infty \alpha_\gamma(t)\,dt\right) = \left[\exp\left(\int_u^\infty \alpha_0(t)\,dt\right)\right]^{\frac{1}{1-\gamma}} = (\phi_0(u))^{1/1-\gamma}$$

Surplus-dependent tax level

$$\Rightarrow \phi_\Gamma(u) = \exp\left(-\frac{\lambda}{c}\int_u^\infty \frac{\mathbb{P}(V_{\max} > t)}{1-\gamma(t)}\,dt\right) = \exp\left(-\int_u^\infty \left(\frac{d}{dt}\log\phi_0(t)\right)\frac{1}{1-\gamma(t)}\,dt\right)$$

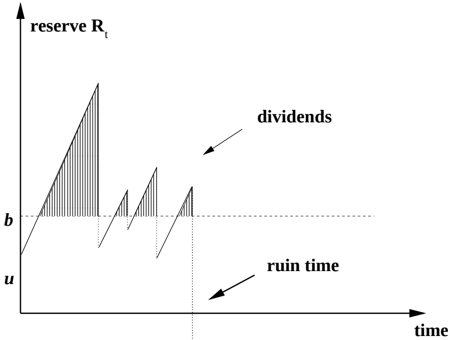
- Extension to Lévy processes, etc.

Expected discounted tax payments

$v(u) = \mathbb{E}\left[\int_0^{\tau} e^{-\delta t}\gamma\, 1_{\{RM(t)\}}\, dt\right]$:

Crucial quantity: $B(u,b) := \mathbb{E}[e^{-\delta\tau^+(u,0,b)}] = \frac{V(u,b)}{V(b,b)}$

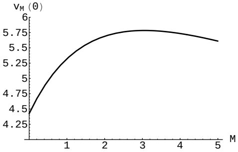
$V(u,b)\ldots$ expected discounted dividends with horizontal barrier strategy



$$v'(u) = \frac{v(u)}{(1-\gamma)V(u,u)} - \frac{\gamma}{1-\gamma} \quad \text{with} \quad v(\infty) = \gamma \lim_{u\to\infty} V(u,u) = \frac{\gamma}{\rho}$$

On from which capital $M > u$ shall authority collect taxes?

Example: $X_i \sim \text{Exp}(1)$, $c = 2$, $\lambda = 1$, $\delta = 0.04$, $\gamma = 0.5$:

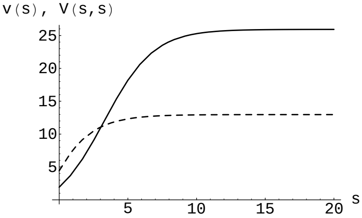


$$v_M(u) = \frac{V(u,M)}{V(M,M)}\, v(M)$$

Optimal M^*: If $v(0) > \frac{c}{\lambda+\delta}$: $\quad v(M^*) = V(M^*, M^*) \qquad \Longrightarrow v_{M^*}(u) = V(u, M^*)$

$$\Longleftrightarrow v'(M^*) = 1$$

If $v(0) \leq \frac{c}{\lambda+\delta}$: $\quad M^* = 0 \qquad \Longrightarrow v_{M^*}(u) = v(u)$



Example: Reinsurance portfolio with few data points $X_1, \ldots, X_n$

typical procedure:

- Estimation of $\mathbb{E}(X)$ by $\hat{\mu} = (X_1 + \cdots + X_n)/n$
- Estimation of $\text{CoV}(X) = \frac{std(X)}{\mathbb{E}(X)}$ from “related” portfolio
- Premium $P = \hat{\mu} + \beta \ \text{CoV}(X)$

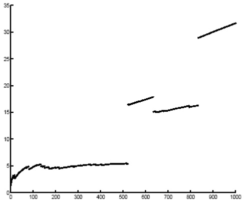
Typical claim distribution:

$$\text{Pareto: } P(X_i > x) = \left(1 + \frac{x}{k}\right)^{-\alpha} \ (\alpha, k > 0)$$

Industry fire: $\hat{\alpha} < 1 \longrightarrow \mathbb{E}(X) = \infty$

Houses fire: $1 < \hat{\alpha} < 2 \longrightarrow \mathbb{E}(X) < \infty$, but $Var(X) = \infty$

$\widehat{\text{CoV}}(X)$ from Pareto(0.5):



Asymptotic Analysis of Measures of Variation

A. & Teugels (2006,2009,2010)

$\{X_i\}_{i=1}^n$ iid pos. r.v. (d.f. F):

$$\boxed{T_n := \frac{X_1^2+X_2^2+\ldots+X_n^2}{(X_1+X_2+\ldots+X_n)^2}}$$

$$n\, T_n = \widehat{\mathrm{CoV}(X)}^2 + 1$$

$1 - F(x) \sim x^{-\alpha}\ell(x), \quad \lim_{x\to\infty} \frac{\ell(tx)}{\ell(x)} = 1\, \forall\ t > 0.$

$\alpha < 2$: DA condition

$$\beta > 0:\ \mathbb{E}(X_1^\beta) = \beta\int_0^\infty x^{\beta-1}(1-F(x))\,dx \begin{cases} < \infty, & \beta < \alpha \\ = \infty, & \beta > \alpha \end{cases}$$

$$\varphi(s) := \int_0^\infty e^{-sx}\,dF(x),\ s \geq 0 \longrightarrow \mathbb{E}\left(\frac{1}{X_1^\beta}\right) = \frac{1}{\Gamma(\beta)}\int_0^\infty s^{\beta-1}\varphi(s)\,ds$$

(Fuchs et al. (2001))

$$\mathbb{E}(T_n) = \mathbb{E}\frac{X_1^2+X_2^2+\ldots+X_n^2}{(X_1+X_2+\ldots+X_n)^2} = n\int_0^\infty s\varphi''(s)\varphi^{n-1}(s)\,ds$$

$$\mathbb{E}\,(T_n^k) = \sum_{r=1}^{k}\ \sum_{k_1,\ldots,k_r\geq 1} \frac{k!}{k_1!\cdots k_r!}\, B(n,k_1,\ldots,k_r) \quad (k\in\mathbb{N})$$

Theorem. *Let $X \in DA(\alpha)$; $(0 < \alpha < 1)$.*

$$\forall\, k \in \mathbb{N}: \lim_{n\to\infty} \mathbb{E}(T_n^k) = const. = \frac{1}{\prod_{\ell=1}^{k}(2\ell-1)} \sum_{j=0}^{k} (-1)^j\, a_{jk}\, \alpha^j,$$

with a_{jk}.. coefficient of $t^j z^k$ in expansion of continued fraction
$$M(t,z) = \cfrac{1}{1 - \cfrac{(t+1)z}{1 - \cfrac{(t+2)z}{1 - \cfrac{(t+3)z}{1 - \cdots}}}}$$

Sketch of Proof: $\varphi^{(m)}(s) \sim (-1)^m \alpha\, \Gamma(m-\alpha)\, s^{\alpha-m} \ell(1/s)$ for $s \downarrow 0$, $\quad m > \alpha$.

$n\, a_n^{-\alpha}\, \ell(a_n)\Gamma(1-\alpha) \to 1 \qquad \Rightarrow \lim_{n\to\infty} \varphi^n(\frac{s}{a_n}) = e^{-s^\alpha} \quad \forall s \geq 0.$

$$B(n, k_1, \ldots, k_r) = \frac{\binom{n}{r}}{a_n \Gamma(2k)} \int_0^\infty \left(\frac{t}{a_n}\right)^{2k-1} \varphi^{(2k_1)}\left(\frac{t}{a_n}\right) \cdots \varphi^{(2k_r)}\left(\frac{t}{a_n}\right) \underbrace{\varphi^{n-r}\left(\frac{t}{a_n}\right)}_{\to\, e^{-t^\alpha}} dt$$

$$\sim \frac{\alpha^r \binom{n}{r}}{a_n \Gamma(2k)} \int_0^\infty \left(\frac{t}{a_n}\right)^{2k-1} \left(\frac{t}{a_n}\right)^{r\,\alpha-2k} \ell^r\left(\frac{a_n}{t}\right) \left(\prod_{j=1}^{r} \Gamma(2k_j-\alpha)\right) e^{-t^\alpha}\, dt$$

$$\sim \frac{\alpha^r \prod_{j=1}^r \Gamma(2k_j-\alpha)}{\Gamma(2k)} \underbrace{\frac{\binom{n}{r}\, \ell^r(a_n)}{a_n{}^{r\,\alpha}}}_{\to\, \Gamma(1-\alpha)^{-r}/r!} \underbrace{\int_0^\infty t^{r\,\alpha-1} e^{-t^\alpha}\, dt}_{=\,(r-1)!/\alpha} \sim \frac{\alpha^{r-1} \prod_{j=1}^r \Gamma(2k_j-\alpha)}{r\,\Gamma(1-\alpha)^r\, \Gamma(2k)}.$$

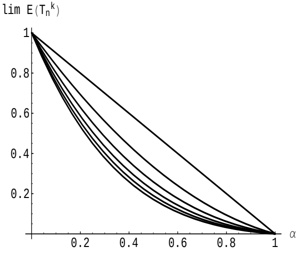
$$\lim_{n\to\infty} \mathbb{E}\,(T_n^k) = \frac{k!}{(2k-1)!} \sum_{r=1}^{k} \frac{\alpha^{r-1}}{r\,\Gamma(1-\alpha)^r} \sum_{\substack{k_1,\ldots,k_r \geq 1 \\ k_1+\ldots+k_r=k}} \prod_{j=1}^{r} \frac{\Gamma(2k_j-\alpha)}{k_j!} \qquad \square$$

$$\lim_{n\to\infty} \mathbb{E}(T_n) \sim 1-\alpha,$$

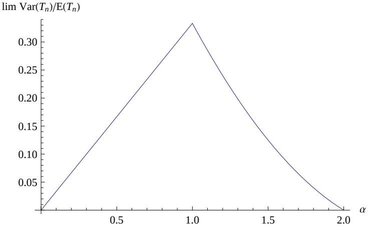
$$\lim_{n\to\infty} \mathbb{E}(T_n^2) \sim \tfrac{1}{3}(1-\alpha)(3-2\,\alpha),$$

$$\lim_{n\to\infty} \mathbb{E}(T_n^3) \sim \tfrac{1}{15}(1-\alpha)(15-17\,\alpha+5\,\alpha^2),$$

$$\lim_{n\to\infty} \mathbb{E}(T_n^4) \sim \tfrac{1}{105}(1-\alpha)(105-155\,\alpha+79\,\alpha^2-14\,\alpha^3),$$



	$\mathbb{E}(T_n)$	$\mathrm{Var}\,(T_n)$
$0<\alpha<1$	$1-\alpha$	$\frac{\alpha(1-\alpha)}{3}$
$\alpha=1$	$\frac{\ell(a_n)}{\tilde{\ell}(a_n)}\ (\to 0)$	$\frac{1}{3}\ \frac{\ell(a_n)}{\tilde{\ell}(a_n)}\ (\to 0)$
$1<\alpha<2$	$\frac{\Gamma(2-\alpha)\Gamma(1+\alpha)}{\mu^2}\ n^{1-\alpha}\ell(n)$	$\frac{\Gamma(4-\alpha)\Gamma(1+\alpha)}{6\mu^{\alpha}}\ n^{1-\alpha}\ell(n)$
$\alpha=2$	$\frac{2}{\mu^2}\ \frac{\tilde{\ell}(n)}{n}$	$\frac{\ell(n)}{3\,n\,\mu^2}$
$2<\alpha<4$	$\frac{\mu_2}{\mu^2\,n}$	$\frac{\Gamma(4-\alpha)\Gamma(1+\alpha)}{6\,\mu^{\alpha}}\ n^{1-\alpha}\ell(n)$
$\alpha\geq 4$	$\frac{\mu_2}{\mu^2\,n}$	$\frac{\mu_4\mu^2-\mu_2^2\mu^2+4\,\mu_2^3-4\mu\mu_2\mu_3}{\mu^6}\ \frac{1}{n^3}$

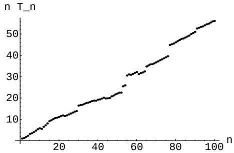


$$\boxed{T_n := \frac{X_1^2+X_2^2+\ldots+X_n^2}{(X_1+X_2+\ldots+X_n)^2}}$$

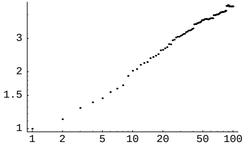
$0 < \alpha < 1$: $\mathbb{E}(T_n) \sim 1-\alpha$

$1 < \alpha < 2$: $\mathbb{E}(T_n) \sim \frac{\Gamma(2-\alpha)\Gamma(1+\alpha)}{\mu^2}\, n^{1-\alpha}\ell(n)$

- Alternative estimator for extreme value index $1/\alpha$



Pareto ($\alpha = 0.5$)



Pareto($\alpha = 1.5$)

- Test for finiteness of $\mathbb{E}(X)$ and Var(X) (for $X \in DA(\alpha)$)