Risk processes with premium adjusted to solvency targets

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Probability of ruin

..."never in the field of scientific inquiry has so much effort been invested in the study of such a small probability!"

”Small probability”

- certain local (in time and space) circumstances where sound actuarial practices and regulations were in place
- risk theory is not very useful in practice

Risk Theory - debate

Academics: "risk theory based on stochastic processes is an indispensable tool in actuarial science and practice" (*Harald Cramer*)

Practicing actuaries: "academic subject... to recherché to be of interest to the general run of actuaries"

Classical collective risk model

The insurance portfolio at time t starting with initial capital u :

$$U(t) = u + ct - \sum_{k=1}^{N(t)} Y_k.$$

- $N(t)$ Poisson/renewal process that counts the nr of claims in $(0, t]$
- $(Y_k)_{k\geq 0}$ denote the iidrv claim sizes $\sim G$
- $(T_k)_{k\geq 0}$, with $\tau_k = T_k - T_{k-1}$ iidrv inter-arrival times
- $(Y_k)_{k\geq 0}$ and $(\tau_k)_{k\geq 1}$ are assumed to be independent.)
- c represents the constant premium rate, $c \geq \lambda \mathbb{E} Y$

Probability of ruin

- Time of ruin =first time the process becomes negative

$$T_u = \inf_{t \geq 0} \{U(t) < 0 \mid U(0) = u\}$$

- Probability of ruin:

$$\psi(u) = \mathbb{P}(T_u < \infty)$$

- Lundberg bound:

$$\psi(u) \leq e^{-Ru}$$

- Adjustment coefficient R: $e^{R\,U(t)}$ martingale

Dispute not settled for a century

- academia: spectacular developments
  - models sufficiently well-structured to allow explicit or asymptotic formulas for the ruin probabilities, for their upper and lower bounds
  - (infinitely) large claims, dependence among claims
  - impact of interest (deterministic or stochastic) under strict stationary assumptions
  - optimal stochastic control allowing investments and reinsurance strategies to be managed dynamically to optimize the ruin probability
- practice: graphics of simulations made in complex real life scenarios

Experience rating

- determine the premium rate based on the history of claims
- premiums are dynamically adjusted in some mechanical manner
- popular topic in 70-es and 80-es

Risk theory: Dubey (1977) assumed the Poisson intensity is unknown and random and that the premium rate at any time is proportional to the Bayes estimate of the intensity.

Dubey, A. (1977) Probabilité de ruine lorsque le paramètre Poisson est ajusté a posteriori. *Bulletin de l'Association des Actuaires Suisses*, **2**, 211-224

This talk

- extension of Dubey's approach

- rather than calculate the probability of ruin for a given reserve u and premium function c, we seek to dynamically adjust the premium to achieve a Lundberg-type upper bound for the probability of ruin.

- restriction/condition: claim distributions that have exponential moments

Paper

- Joint work with Veronique Maume-Deschamps and Ragnar Norberg, ISFA, University of Lyon 1, Lyon, France
- To appear in the European Actuarial Journal

Premium

" Ah - so it's the compensator!"

Kiyoshi Itô's spontaneous reaction when once told by an actuary what the insurance premium is. *(Hans Bühlmann)*

The premium should compensate the claims expenses that can be expected in view of past experience.

Compensator

- Marked point process (T_i, Y_i) with associated random measure N

$$\sum_{i:T_i \le t} Y_i = \int_0^t \int_0^\infty y N(ds, dy)$$

- Random measure ν is the compensator of N, if for each Borel set $\mathcal{Y}$,

$$N([0,t]x\mathcal{Y}) - \nu([0,t]x\mathcal{Y})$$

is a martingale

Heuristic interpretation

For $\mathcal{F}_{t-}$ the strictly past history at time t,

$$\nu(dt, dy) = \mathbb{E}[N(dt, dy)|\mathcal{F}_{t-}]$$

We assume the claims are intensity driven,

$$\nu(dt, dy) = \lambda_t dt G_t(dy)$$

i.e. conditional on $\mathcal{F}_{t-}$

- λ_t is the expected number of claims per time unit
- G_t is the claim size distribution at time t

Classical model

$$\nu(dt, dy) = \lambda dt \, G(dy)$$

$$c = (1 + \rho) \, \lambda \, \mathbb{E} Y$$

Lundberg (1903): For claim distributions G with m_G finite in an interval above 0, if there is an $R > 0$ such that

$$e^{R \, U(t)} \quad \text{martingale} \iff c = \lambda \, \frac{m_G(R) - 1}{R}$$

then

$$\psi(u) \leq e^{-Ru}$$

for all u.

Dubey's model

$$\nu(dt, dy) = \Theta dt\, G(dy)$$

$$c_{t} = (1+\rho)\, \lambda_t^N \, \mathbb{E} Y$$

- $\Theta \sim \Gamma(\alpha, \beta)$
- $\lambda_t^N = \mathbb{E}[\Theta | N_t] = \frac{N_t + \alpha}{t + \beta}$ posterior mean of the mixing variable Θ

$\psi(u)$ the same as in the classical model

Our extension

Let $\Lambda_t^N = \int_0^t \lambda_s^N ds$ operational time.

Theorem. Assume that the claim amounts Y_i are iid and independent of $(N_t)_{t\geq 0}$, that $\Lambda_t^N < \infty$ for $t < \infty$ and $\Lambda_\infty^N = \infty$ and for each non-negative and finite constant τ, the stopping time $\inf\{t; \Lambda_t^N \geq \tau\}$ is bounded. Then, if the premium rate is

$$c_t = (1+\rho)\,\lambda_t^N\,\mathbb{E}Y$$

with $\rho > 0$ constant, the probability of ruin is independent of the form of intensity λ^N.

Remarks

- $\psi(u)$ depends only on the safety loading ρ and the claim size distribution
- arbitrary intensity process, iid claims independent of the claim counting process

- for the claim sizes the only assumption necessary is the existence of the adjustment coefficient R
- Lundberg's upper bound valid

Solvency control

- Question: what initial reserve u must be provided in order that

$$\psi(u) \leq \varepsilon?$$

- Answer: when claims have exponential moments, $\psi(u) \leq e^{-Ru}$, implying

$$u = -\frac{\ln(\varepsilon)}{R}$$

Another solvency control

- Question: for a given claim process Y (company's liabilities) and a given initial reserve u (company's financial capacity) what premium rate c_t is sufficient to meet the solvency requirement

$$\psi(u) \leq \varepsilon?$$

- Answer: Adjust the premium to produce a pre-specified value

$$R = -\frac{\ln(\varepsilon)}{u}$$

Theorem

Assume that there exists an $R > 0$ such that $m_{G_t^N}(R)$ is finite for all t almost surely. If the premium rate at any time t is chosen as

$$c_t = \frac{\lambda_t^N}{R} \int_0^\infty \left(e^{Ry} - 1\right) G_t^N(dy) \quad (*)$$

then the Lundberg upper bound holds true.

Remark

The result suggests a reason why the adjustment coefficient deserves its name:

- starting from a pre-specified probability of ruin $\psi(u)$
- one determines $R = -\frac{\ln(\varepsilon)}{u}$
- then adjusts continually the premium to R through $(*)$

Proof

- Liability process

$$X_t = \int_0^t \int_0^\infty y\, N(dt, dy) - \int_0^t c_s\, ds$$

- Define the process $M = (M_t)_{t \geq 0}$ by

$$M_t \;=\; e^{R\,X_t} \;=\; e^{R\left(\int_0^t \int_0^\infty y\, N(dt,dy) - \int_0^t c_s\, ds\right)} .$$

- The idea is to choose c such that M becomes a martingale.

Proof

- Applying optional stopping to the martingale M and the bounded stopping time $T_u \wedge n$, where $n > 0$ is a constant, we have

$$1 = M_0 = \mathbb{E}\left[e^{R X_{T_u \wedge n}}\right] \geq \mathbb{E}\left[e^{R X_{T_u \wedge n}} 1[T_u \leq n]\right] \geq \mathbb{E}\left[e^{R u} 1[T_u \leq n]\right],$$

hence

$$1 \geq e^{R u} \mathbb{P}[T_u \leq n]$$

- Upon sending n to ∞, one obtains the claimed Lundberg inequality.

Proof

It remains to prove that the premium function $(*)$ serves to martingalize M.

- By the general chain rule (Itô's formula), the dynamics of M is

$$\begin{aligned} dM_t &= M_{t-}\left[\int_0^\infty \left(e^{Ry}-1\right) N(dt,dy) - R\,c_t\,dt\right] \\ &= M_{t-}\int_0^\infty \left(e^{Ry}-1\right)\left[N(dt,dy) - \lambda^N\,dt\,G_t^N(dy)\right] \\ &\quad + M_{t-}\underbrace{\left[\lambda_t^N \int_0^\infty \left(e^{Ry}-1\right)\, G_t^N(dy) - R\,c_t\right]}_{drift} dt\,. \end{aligned}$$

- In order that M be a martingale, the drift term must be null.

Practical implementation and problems arising

- The implementation of the idea works for very general models.
- The calculations and computational effort required depend on the model assumptions.

- Maybe surprisingly, a major issue is to find parametric models that are both realistic and mathematically tractable and allow dependence between waiting times and amounts.
- A natural way is to start from marginal distributions with desired properties and to introduce dependence through a copula

"Ah – so it is the compensator!"

- This piece of common sense is not reflected in the bulk of ruin theory, a collection of results on crossing probabilities for processes with nice stationarity properties hence constant premium rate.
- In practical risk management one continually regulates premiums with a view to statistical evidence and solvency requirements, business objectives, market conditions, etc.
- Here we have argued that the premium should be adapted to past experience, with a safety loading added to the compensator in such a manner as to meet a requirement on the probability of ultimate ruin.

Thank you for your attention!