

# Explicit ruin formulas for models with dependence among risks

Hansjörg Albrecher <sup>a</sup>, Corina Constantinescu <sup>a</sup>, Stephane Loisel <sup>b</sup>

<sup>a</sup>Department of Actuarial Science, Faculty of Business and Economics, University of Lausanne,  
Batiment Extranef, UNIL-Dorigny, CH-1015 Lausanne, Switzerland.

<sup>b</sup>Université de Lyon, Université Lyon 1, Institut de Science Financière et d'Assurances,  
50 Avenue Tony Garnier, F-69007 Lyon, France.

---

## Abstract

We show that a simple mixing idea allows to establish a number of explicit formulas for ruin probabilities and related quantities in collective risk models with dependence among claim sizes and among claim inter-occurrence times. Examples include compound Poisson risk models with completely monotone marginal claim size distributions that are dependent according to Archimedean survival copulas as well as renewal risk models with dependent inter-occurrence times.

---

**Keywords:** Ruin probability, frailty models, mixing, Archimedean copulas, completely monotone distributions.

## 1. Introduction

Assume that the free surplus of an insurance company at time  $t$  is modeled by

$$R(t) = u + ct - \sum_{k=1}^{N(t)} X_k, \quad (1)$$

where  $u$  is the initial surplus, the stochastic process  $N(t)$  denotes the number of claims up to time  $t$  and the random variables  $X_k$  refer to the corresponding claim amounts. Here  $c > 0$  is a constant premium intensity and  $u \geq 0$  is the initial surplus in the portfolio. The classical Cramér-Lundberg risk model assumes that  $N(t)$  is a homogeneous Poisson process with intensity  $\lambda$ , which is independent of the claim sizes and the claim sizes are independent and identically distributed. Under this assumption, various quantities can be calculated explicitly for certain classes of claim size distributions, including the probability of ruin  $\psi(u) = P(R(t) < 0 \text{ for some } t > 0)$ . However, the independence assumption can be too restrictive in practical applications and it is natural to look for explicit formulas for  $\psi(u)$  and related quantities in the presence of dependence among the

---

*Email address:* [hansjoerg.albrecher@unil.ch](mailto:hansjoerg.albrecher@unil.ch), [corina.constantinescu@unil.ch](mailto:corina.constantinescu@unil.ch),  
[stephane.loisel@univ-lyon1.fr](mailto:stephane.loisel@univ-lyon1.fr) (Hansjörg Albrecher <sup>a</sup>, Corina Constantinescu <sup>a</sup>, Stephane Loisel <sup>b</sup>)

risks. Over the last decades a number of dependence structures have been identified that allow for analytical formulas (see e.g. Chapter XIII of Asmussen and Albrecher (2010) for a recent survey), but an exhaustive identification of this set of models is far from complete.

The purpose of this paper is to provide an additional class of dependence models for which explicit expressions for  $\psi(u)$  can be obtained. To that end, we start with simpler models for which explicit solutions are available and subsequently mix over involved parameters. This changes the marginal distributions and introduces dependence among the risks. One can then balance the marginal distribution of the risks with their dependence structure in such a way that properties like level crossing probabilities can be studied without direct treatment of the dynamics of the process. In other words, the mixing of the parameters can be carried over to the mixing of the final quantities under study. This results in a new set of dependence models for which explicit results can be obtained and may serve as a skeleton within larger model classes. For transparency of exposition, the analysis of the paper will be in terms of the ruin probability  $\psi(u)$ , but the principle can be applied to many other quantities and also to applications beyond risk theory.

We would like to note that mixing over parameter values of the risk process is a classical tool (see for instance Bühlmann (1972), who considered this in the context of credibility-based dynamic premium rules). In this paper we want to suggest such mixing procedures as a general tool of dependence modeling in collective risk theory, as its potential does not seem to be sufficiently explored yet.

The rest of the paper is organized as follows. Sections 2 and 3 work out the ideas in detail and show how the Archimedean dependence structure naturally enters in this approach. Examples for explicit ruin probability formulas are given. Section 4 then highlights a number of further possible extensions of the method and Section 5 concludes.

## 2. Compound Poisson models with completely monotone claim sizes and Archimedean dependence

Let  $\Theta$  be a positive random variable with cdf  $F_\Theta$  and consider the classical compound Poisson risk model (1) with exponential claim sizes that fulfill, for each  $n$ ,

$$P(X_1 > x_1, \dots, X_n > x_n \mid \Theta = \theta) = \prod_{k=1}^n e^{-\theta x_k}. \quad (2)$$

That is, given  $\Theta = \theta$ , the  $X_k$  ( $k \geq 1$ ) are conditionally independent and distributed as  $Exp(\theta)$ . However, the resulting marginal distributions of the  $X_k$ 's will now in general not be exponential any more and the claim sizes will be dependent. Let  $\psi_\theta(u)$  denote the ruin probability of the classical compound Poisson risk model with independent  $Exp(\theta)$  claim amounts given by

$$\psi_\theta(u) = \min \left\{ \frac{\lambda}{\theta c} \exp \left\{ -\left( \theta - \frac{\lambda}{c} \right) u \right\}, 1 \right\}, \quad u \geq 0. \quad (3)$$

Then, for the dependence model (2), the ruin probability is given by

$$\psi(u) = \int_0^\infty \psi_\theta(u) dF_\Theta(\theta). \quad (4)$$

Since for  $\theta \leq \theta_0 = \lambda/c$  the net profit condition is violated and consequently  $\psi_\theta(u) = 1$  for all  $u \geq 0$ , this can be rewritten as

$$\psi(u) = F_\Theta(\theta_0) + \int_{\theta_0}^\infty \psi_\theta(u) dF_\Theta(\theta). \quad (5)$$

An immediate consequence is that in this dependence model

$$\lim_{u \rightarrow \infty} \psi(u) = F_\Theta(\theta_0), \quad (6)$$

which is positive whenever the random variable  $\Theta$  has probability mass at or below  $\theta_0 = \lambda/c$  (and the latter is useful for the construction of dependence models as will be seen below).

**Proposition 2.1.** *The dependence model characterized by (2) can equivalently be described by having marginal claim sizes  $X_1, X_2, \dots$  that are completely monotone, with a dependence structure due to an Archimedean survival copula with generator  $\phi = (\widetilde{F}_\Theta)^{-1}$  for each subset  $(X_{j_1}, \dots, X_{j_n})$  (for  $j_1, \dots, j_n$  pairwise different), where  $\widetilde{F}_\Theta$  denotes the Laplace-Stieltjes transform of  $F_\Theta$ .*

**Proof.** The proof is essentially due to Oakes (1989), who considered the case  $n = 2$ , see also Denuit et al. (2005, p.229). In order to keep the paper self-contained, we give here a rather direct line of reasoning. Due to (2), for each  $n$ , the joint distribution of the tail of  $X_1, \dots, X_n$  can be written as

$$P(X_1 > x_1, \dots, X_n > x_n) = \int_0^\infty e^{-\theta(x_1 + \dots + x_n)} dF_\Theta(\theta) = \widetilde{F}_\Theta(x_1 + \dots + x_n). \quad (7)$$

At the same time, the representation with survival copula  $\widehat{C}$  is given by

$$P(X_1 > x_1, \dots, X_n > x_n) = \widehat{C}(\overline{F}_X(x_1), \dots, \overline{F}_X(x_n)),$$

where  $\overline{F}_X(x_i) = 1 - F_X(x_i)$  is the tail of the marginal claim size  $X_i$  (note that the  $X_i$ 's are all identically distributed). If the survival copula is Archimedean with generator  $\phi$ , then it admits the representation  $\widehat{C}(\overline{F}_X(x_1), \dots, \overline{F}_X(x_n)) = \phi^{-1}(\phi(\overline{F}_X(x_1)) + \dots + \phi(\overline{F}_X(x_n)))$ , which due to

$$\overline{F}_X(x_i) = \int_0^\infty e^{-\theta x_i} dF_\Theta(\theta) = \widetilde{F}_\Theta(x_i), \quad i = 1, \dots, n, \quad (8)$$

exactly matches with (7) when the generator is chosen to be  $\phi(t) = (\widetilde{F}_\Theta)^{-1}(t)$ . Note that as the inverse of a Laplace-Stieltjes transform of a cdf,  $\phi$  is a continuous strictly decreasing function from  $[0,1]$  to  $[0, \infty]$  with  $\phi(0) = \infty$  and  $\phi(1) = 0$  and  $\phi^{-1}$  is completely monotone, so that the Archimedean copula is well-defined for all  $n$  (see e.g. Nelsen (1999, Th.4.6.2)). At the same time, from (8) one sees that the marginal random variables  $X_i$  are necessarily completely monotone.  $\diamond$

**Remark 2.2.** The above mixing construction may be thought of as sampling a realisation  $\theta$  of  $\Theta$  according to  $F_\Theta$  and then following a trajectory of the usual independent risk model with parameter  $\theta$ , so that the dependence is introduced through the common realized value of  $\theta$  among all possible values from the positive halfline. Correspondingly, the resulting dependence will be stronger the more spread out the distribution of  $\Theta$  is. However, this mixing construction is just the tool to establish the explicit formula (3), but need not be the causal reason for the dependence model. Having established this result, one may now equivalently start in two ways: For any risk model with completely monotone marginal claim size random variable  $X_i$ , there is an expression of the form (8) for some positive random variable  $\Theta$  and then formula (3) holds for a dependence structure with Archimedean survival copula and generator  $\phi = (\bar{F}_\Theta)^{-1}$ . Alternatively, one may start with specifying the Archimedean survival copula through its generator and then the above relations give a corresponding marginal distribution for which the explicit formula (3) holds. Note that for each choice of the generator  $\phi$ , Kendall's tau for each bivariate pair is given by  $\tau_0 = 1 + 4 \int_0^1 \phi(t)/\phi'(t) dt$ .

Let us now look at some particular examples.

**Example 2.3** (Pareto claims with Clayton copula dependence).

If  $\Theta$  is *Gamma*( $\alpha, \beta$ ) distributed with density

$$f_\Theta(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \quad \theta > 0,$$

the resulting mixing distribution for the marginal claim size  $X_k$  is

$$\bar{F}_X(x) = \int_0^\infty e^{-\theta x} f_\Theta(\theta) d\theta = \left(1 + \frac{x}{\beta}\right)^{-\alpha}, \quad x \geq 0$$

(see e.g. Klugman et al. (2008)). That is, the  $X_k$ 's are *Pareto*( $\alpha, \beta$ ) distributed and due to Proposition 2.1 they follow a dependence structure according to an Archimedean survival copula with generator

$$\phi(t) = t^{-1/\alpha} - 1,$$

which is the Clayton copula with parameter  $\alpha$  (see also Yeh (2007)). Consequently, the upper tail dependence index between two claim amounts is

$$\lambda_U = 2 - 2^{-1/\alpha}$$

(the asymptotic behavior of  $P(X_1 + X_2 > x)$  in this situation is for instance studied in Alink et al. (2004)). From (5) it now follows that for this model

$$\psi(u) = 1 - \frac{\Gamma(\alpha, \beta\theta_0)}{\Gamma(\alpha)} + \theta_0 e^{\theta_0 u} \beta \left(1 + \frac{u}{\beta}\right)^{-(\alpha-1)} \frac{\Gamma(\alpha-1, (\beta+u)\theta_0)}{\Gamma(\alpha)},$$

where  $\Gamma(\alpha, x) = \int_x^\infty w^{\alpha-1} e^{-w} dw$  is the incomplete Gamma function and  $\theta_0 = \lambda/c$ . In particular (also from (6)),

$$\lim_{u \rightarrow \infty} \psi(u) = 1 - \frac{\Gamma(\alpha, \frac{\beta\lambda}{c})}{\Gamma(\alpha)}. \quad (9)$$

Using  $\lim_{x \rightarrow \infty} \frac{\Gamma(s, x)}{x^{s-1}e^{-x}} = 1$ , one can further deduce that the convergence towards this constant is of asymptotic order  $u^{-1}$ . Finally, for  $u = 0$  we obtain the simple formula

$$\psi(0) = 1 - \frac{\Gamma(\alpha, \beta\theta_0)}{\Gamma(\alpha)} + \beta\theta_0 \frac{\Gamma(\alpha - 1, \beta\theta_0)}{\Gamma(\alpha)}.$$

◇

**Example 2.4** (Weibull claims with Gumbel copula dependence).

If  $\Theta$  is stable (1/2) distributed (also called Lévy distributed) with density

$$f_{\Theta}(\theta) = \frac{\alpha}{2\sqrt{\pi}\theta^3} e^{-\alpha^2/4\theta}, \quad \theta > 0,$$

then the resulting marginal distribution tail of the claim size random variable  $X$  is

$$\bar{F}_X(x) = \int_0^{\infty} e^{-\theta x} f_{\Theta}(\theta) d\theta = \exp\{-\alpha x^{1/2}\}, \quad x \geq 0,$$

so that the claim sizes are Weibull distributed with shape parameter 1/2. Since  $\tilde{F}_{\Theta}(s) = e^{-\alpha\sqrt{s}}$ , one obtains that the generator of the Archimedean copula is in this case given by  $\phi(t) = (-\ln t)^2$  (for all values of  $\alpha$ ). Hence the underlying survival copula is a particular Gumbel copula, and consequently the claim sizes are asymptotically independent. Note that the marginal distribution varies according to the choice of  $\alpha$ , whereas the copula stays invariant.

From (5) we now get

$$\psi(u) = F_{\Theta}(\lambda/c) + \int_{\lambda/c}^{\infty} \frac{\lambda}{\theta c} e^{-\theta u} e^{\lambda c/u} \frac{\alpha}{2\sqrt{\pi}\theta^3} e^{-\alpha^2/4\theta} d\theta,$$

which can be expressed through the error function  $\text{Erfc}(z) = 1 - \text{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-w^2} dw = 2\Phi(-z\sqrt{2})$  as

$$\begin{aligned} \psi(u) = \text{Erfc}\left(\frac{\alpha}{2\sqrt{\lambda/c}}\right) + \frac{\lambda}{\alpha^2 c} e^{-\alpha^2 c/4\lambda} \left[ -\frac{2\alpha}{\sqrt{\pi}\lambda/c} + e^{(c\alpha-2\sqrt{u}\lambda)^2/4c\lambda} (1+\alpha\sqrt{u}) \text{Erfc}\left(\sqrt{u\lambda/c} - \frac{\alpha}{2\sqrt{\lambda/c}}\right) \right. \\ \left. + e^{(c\alpha+2\sqrt{u}\lambda)^2/4c\lambda} (-1+\alpha\sqrt{u}) \text{Erfc}\left(\sqrt{u\lambda/c} + \frac{\alpha}{2\sqrt{\lambda/c}}\right) \right]. \end{aligned}$$

Note that for  $z > 0$ ,  $\text{Erfc}(z) = \Gamma(1/2, z^2)/\sqrt{\pi}$ . For  $u = 0$  we have

$$\psi(0) = \text{Erfc}\left(\frac{\alpha}{2\sqrt{\lambda/c}}\right) - \frac{2\sqrt{\lambda/c}}{\alpha\sqrt{\pi}} e^{-c\alpha^2/4\lambda} + \frac{2\lambda}{c\alpha^2} \text{Erf}\left(\frac{\alpha}{2\sqrt{\lambda/c}}\right)$$

and

$$\lim_{u \rightarrow \infty} \psi(u) = \text{Erfc}\left(\frac{\alpha}{2\sqrt{\lambda/c}}\right). \quad (10)$$

◇

Whenever  $F_{\Theta}(a) = 0$  for some  $a > 0$  (i.e. there is no probability mass in the neighborhood of the origin), the resulting marginal claim size distribution will be light-tailed. The following example illustrates this for a resulting marginal (completely monotone) Gamma distribution.

**Example 2.5** (Dependent Gamma claims).

Since the  $\text{Gamma}(\alpha, \beta)$  distribution with shape parameter  $\alpha \leq 1$  is completely monotone, it can be obtained as a marginal claim distribution with the above method. Correspondingly, if

$$\bar{F}_X(x_i) = \Gamma(\alpha, \beta x_i) / \Gamma(\alpha) \quad \text{with } \alpha < 1, \quad (11)$$

then due to (8) we need to identify the mixing distribution by inverse Laplace transformation. This can be done by complex analysis techniques in an analogous way as in Albrecher and Kortschak (2009) and one arrives at the mixing density

$$f_{\Theta}(\theta) = \frac{\sin(\alpha\pi)}{\pi\theta} \left( \frac{\theta}{\beta} - 1 \right)^{-\alpha}, \quad \theta > \beta,$$

i.e. the mixing distribution has a Pareto-type tail. Formula (4) now gives the ruin probability for a risk model with marginal claim size distribution (11) and Archimedean dependence structure among claims with generator  $\phi(t)$ , which is the inverse with respect to  $x$  of the normalized incomplete Gamma function  $\Gamma(\alpha, \beta x) / \Gamma(\alpha)$ . Although this generator can not be determined explicitly, it is clear that it does not depend on  $\beta$  (since the generator is invariant with respect to multiplication by a constant). Correspondingly, varying over  $\beta$ , (4) gives an explicit representation of  $\psi(u)$  for a whole class of marginal Gamma distributions with the same dependence structure. The result can be compared with the exact ruin probability for the same marginal claim distribution, but with independent claims (see e.g. Grandell (1992, p.14)).  $\diamond$

Note also that if the support of  $\Theta$  is the interval  $[\theta_1, +\infty)$  for a  $\theta_1 > \theta_0$ , then the probability of ruin  $\psi(u)$  decays exponentially fast with  $u$  and is upper-bounded by  $\psi_{\theta_1}(u)$ , where  $\psi_{\theta_1}(u) = \frac{\lambda}{\theta_1 c} \exp\{-(\theta_1 - \frac{\lambda}{c})u\}$  is the ruin probability of the independent Cramér-Lundberg model with parameter  $\theta_1$ .

**Remark 2.6.** From the general construction of Archimedean copulas (see e.g. Joe (1997, p.86)), it is clear that the principle outlined in this section can be extended to conditionally independent tails beyond the used exponential tails in (2), still preserving the Archimedean dependence structure. In particular, whenever the conditional tail of the marginals can be written in the power form  $P(X_i > x_i | \Theta = \theta) = (\bar{G}(x_i))^{\theta}$  for some distribution function  $G(x)$  and

$$P(X_1 > x_1, \dots, X_n > x_n | \Theta = \theta) = \prod_{k=1}^n (\bar{G}(x_k))^{\theta}$$

for all  $n$ , i.e.  $\theta$  is the common mixing parameter, then

$$\begin{aligned}
P(X_1 > x_1, \dots, X_n > x_n) &= \int_0^\infty [\bar{G}(x_1)]^\theta \cdots [\bar{G}(x_n)]^\theta dF_\Theta(\theta) \\
&= \tilde{F}_\Theta(-\log \bar{G}(x_1) - \cdots - \log \bar{G}(x_n)) \\
&= \tilde{F}_\Theta(\tilde{F}_\Theta^{-1}(\bar{F}_X(x_1)) + \cdots + \tilde{F}_\Theta^{-1}(\bar{F}_X(x_n))).
\end{aligned}$$

Hence, the dependence structure is again Archimedean with generator  $\phi(t) = (\tilde{F}_\Theta)^{-1}(t)$ , where now  $\bar{F}_X(x_i) = \tilde{F}_\Theta(-\log \bar{G}(x_i))$  (in the context of survival analysis,  $\theta$  is often interpreted as the frailty parameter). One concrete example would be to choose the claim size distribution in the independent risk model to be Pareto( $\alpha, \beta$ ) distributed, where  $\alpha$  would now be the mixing parameter  $\theta$  (so that  $\bar{G}(x)$  would be the tail of a Pareto(1,  $\beta$ ) random variable). Then (4) applies and one still has the freedom to choose the mixing distribution of  $\Theta$  to identify a number of formulas for different dependence structures and marginals. However, this is only of limited usefulness, as for Pareto distributed claims there is no fully explicit formula for  $\psi_\theta(u)$  available (but see Ramsay (2003) and Albrecher and Kortschak (2009) for integral representations of  $\psi_\theta(u)$ ).

**Remark 2.7.** In fact, the construction of dependence through a common factor is not as restrictive as it may appear at first sight: due to a generalization of De Finetti's Theorem by Bühlmann (1960), an exchangeable family of random variables (and exchangeability may be seen as a natural assumption in the risk model context) can in great generality be generated as a mixture over a common parameter of an i.i.d. sequence (see also Feller (1970, p.229)).

### 3. Renewal risk models with completely monotone inter-occurrence time distributions and Archimedean dependence

In much the same way as in Section 2, one can also use ruin probability formulas of the Cramér-Lundberg risk model (which are explicit for certain classes of claim size distributions, see e.g. Asmussen and Albrecher (2010)) and mix over the Poisson parameter  $\lambda$ . If the mixing cdf is denoted by  $F_\Lambda$ , then the resulting ruin probability in the new dependence model is

$$\psi(u) = \int_0^\infty \psi_\lambda(u) dF_\Lambda(\lambda), \quad u \geq 0. \quad (12)$$

Note that this formula was already given by Bühlmann (1972, Eq. (3)), where the mixing procedure was used in the context of dynamic credibility-based premiums for the risk process (see also Dubey (1977), Bühlmann and Gerber (1978) and Gerber (1979, Section 5 of Chapter 9)). Proposition 2.1 now still holds by replacing  $\theta$  with  $\lambda$ , i.e. the resulting dependence structure between the inter-occurrence times  $T_1, T_2, \dots$  (and each subset of them) will be described by an Archimedean copula with generator  $\phi = (\tilde{F}_\Lambda)^{-1}$ , where  $\tilde{F}_\Lambda$  is the Laplace-Stieltjes transform of  $F_\Lambda$ . Note that the resulting (dependent) inter-occurrence times are marginally not exponential any more, but are completely monotone with distribution tail  $P(T_i > t) = \tilde{F}_\Lambda(t)$ . Now the net profit condition will be violated

whenever the realisation of  $\Lambda$  is larger than the threshold value  $\lambda_0 = c/E(X_i)$ . Hence, a refined version of (12) is

$$\psi(u) = \int_0^{\lambda_0} \psi_\lambda(u) dF_\Lambda(\lambda) + \bar{F}_\Lambda(\lambda_0), \quad u \geq 0.$$

Correspondingly,

$$\lim_{u \rightarrow \infty} \psi(u) = \bar{F}_\Lambda(\lambda_0),$$

which is positive whenever  $\Lambda$  has a positive probability to be larger or equal to  $\lambda_0$ .

**Example 3.1** (Pareto inter-occurrence times with Clayton copula dependence).

As in Example 2.3, if  $\Lambda$  is *Gamma*( $\alpha, \beta$ ) distributed, the resulting mixing distribution for the marginal inter-occurrence times  $T_k$  is Pareto distributed with tail

$$\bar{F}_T(t) = \int_0^\infty e^{-\lambda t} f_\Lambda(\lambda) d\lambda = \left(1 + \frac{t}{\beta}\right)^{-\alpha}, \quad t \geq 0.$$

and their dependence structure is described by a Clayton survival copula with generator

$$\phi(t) = t^{-1/\alpha} - 1.$$

Note that at the same time it is a classical result that the number  $N(t)$  of claims up to a fixed time point  $t$  under the Gamma mixing distribution follows a negative binomial distribution.

If one considers the special case of exponentially distributed claim sizes with (now fixed!) parameter  $\theta$ , then clearly

$$\psi_\lambda(u) = \min\left\{\frac{\lambda}{\theta c} \exp\{-(\theta - \frac{\lambda}{c})u\}, 1\right\}, \quad u \geq 0.$$

With (12) and  $\lambda_0 = c\theta$ , this leads to the explicit formula

$$\psi(u) = \frac{\beta^\alpha e^{-\theta u}}{\theta c} (\beta - u/c)^{-1-\alpha} \left(\alpha - \frac{\Gamma(\alpha + 1, c\theta\beta - \theta u)}{\Gamma(\alpha)}\right) + \frac{\Gamma(\alpha, \beta c\theta)}{\Gamma(\alpha)}, \quad u \geq 0. \quad (13)$$

In particular, we have

$$\psi(0) = \frac{1}{\beta c\theta} \left(\alpha - \frac{\Gamma(\alpha + 1, c\theta\beta)}{\Gamma(\alpha)}\right) + \frac{\Gamma(\alpha, \beta c\theta)}{\Gamma(\alpha)}$$

and

$$\lim_{u \rightarrow \infty} \psi(u) = \frac{\Gamma(\alpha, \beta c\theta)}{\Gamma(\alpha)}.$$

In view of  $\lim_{x \rightarrow \infty} \Gamma(s, x)/(x^{s-1}e^{-x}) = 1$ , the convergence towards this constant is again of asymptotic order  $u^{-1}$ .  $\diamond$

**Example 3.2** (Weibull inter-occurrence times with Gumbel copula dependence). As in Example 2.3, if  $\Lambda$  is stable (1/2) distributed with density

$$f_{\Lambda}(\lambda) = \frac{\alpha}{2\sqrt{\pi}\lambda^3} e^{-\alpha^2/4\lambda}, \quad \lambda > 0,$$

then the resulting marginal distribution tail of the inter-occurrence times  $T_k$  is

$$\bar{F}_T(t) = \int_0^{\infty} e^{-\lambda t} f_{\Lambda}(\lambda) d\lambda = \exp\{-\alpha t^{1/2}\}, \quad t \geq 0.$$

That is, the inter-occurrence times are Weibull distributed with shape parameter 1/2 and their dependence structure is described by a Gumbel survival copula with generator  $\phi(t) = (-\ln t)^2$ . From (12), the ruin probability for such a model is given by

$$\begin{aligned} \psi(u) = \frac{\alpha i e^{-i\alpha \sqrt{u/c} - u\theta}}{4\theta \sqrt{cu}} & \left( -1 + \operatorname{Erf}\left(\frac{\alpha}{2\sqrt{c\theta}} - i\sqrt{u\theta}\right) \right. \\ & \left. + e^{2i\sqrt{u/c}\alpha} \operatorname{Erfc}\left(\frac{\alpha}{2\sqrt{c\theta}} + i\sqrt{u\theta}\right) \right) + \operatorname{Erfc}\left(\frac{\alpha}{2\sqrt{c\theta}}\right), \end{aligned} \quad (14)$$

where  $i = \sqrt{-1}$ . It can be shown that the resulting imaginary part of the right-hand side of (14) is zero, so the expression is indeed real. For  $u = 0$  we obtain the limit

$$\psi(0) = \left(1 - \frac{\alpha^2}{2c\theta}\right) \operatorname{Erfc}\left(\frac{\alpha}{2\sqrt{c\theta}}\right) + \frac{\alpha}{\sqrt{\theta\pi c}} e^{-\alpha^2/4c\theta}$$

and for large  $u$  we have  $\lim_{u \rightarrow \infty} \psi(u) = \operatorname{Erfc}\left(\frac{\alpha}{2\sqrt{c\theta}}\right)$ . ◇

**Remark 3.3.** *Since for heavy-tailed inter-occurrence times the usual techniques such as exploiting certain Markovian conditioning arguments do not work, there does not seem to be any explicit formula for  $\psi(u)$  available in the literature. Consequently, formulas (13) and (14) may be the first instances of an explicit ruin probability formula for heavy-tailed identically distributed inter-occurrence times, albeit dependent according to a Clayton and Gumbel copula, respectively.*

#### 4. Further extensions of the method

The mixing idea outlined in this paper can be developed further in many directions. We list here a couple of examples.

**Example 4.1** (Independent parallel mixing). One can mix both inter-occurrence times and claim sizes independently at the same time, leading to the above-described Archimedean copula structure for each. The ruin probability is then

$$\psi(u) = \int_0^{\infty} \int_0^{\infty} \psi_{(\theta,\lambda)}(u) dF_{\Theta}(\theta) dF_{\Lambda}(\lambda),$$

where  $\psi_{(\theta,\lambda)}$  is the conditional probability of ruin given that  $\Theta = \theta$  and  $\Lambda = \lambda$ . Whenever there is an explicit expression for  $\psi_{(\theta,\lambda)}$ , this leads to an explicit expression for  $\psi(u)$  in renewal models with both dependent inter-occurrence times and dependent claim sizes.  $\diamond$

**Example 4.2** (Comonotonic mixing). One can also consider Archimedean dependence between inter-occurrence times and claim sizes and at the same time dependence among claim sizes and among inter-occurrence times. One way to do this is comonotonic mixing, where the realization  $\lambda$  of  $\Lambda$  is a deterministic function of the realisation  $\theta$  of  $\Theta$  in the form

$$\lambda(\theta) = F_{\Lambda}^{-1}(F_{\Theta}(\theta)).$$

The ruin probability in this model then is

$$\psi(u) = \int_0^{\infty} \psi_{(\theta,\lambda(\theta))}(u) dF_{\Theta}(\theta),$$

where  $\psi_{(\theta,\lambda)}$  is the conditional probability of ruin given that  $\Theta = \theta$  and  $\Lambda = \lambda$ .  $\diamond$

**Example 4.3** (Independent light-tailed and dependent heavy-tailed claims). Recall that in Example 2.3, in order to obtain a ruin probability formula for a certain (completely monotone) marginal claim distribution, the dependence structure needed to be fixed. There is in fact a way to vary the dependence structure while leaving the asymptotic tail of the marginal claim distribution unchanged and still receive explicit formulas. Let the surplus process be given by

$$R(t) = u + ct - \sum_{k=1}^{N(t)} X_k - \sum_{l=1}^{N'(t)} Y_l,$$

where there are two independent Poisson processes:  $N'(t)$  with intensity  $\lambda'$  generates (independent)  $Exp(\nu)$  claims  $Y_1, Y_2, \dots$ , where  $\nu$  is a fixed constant. On the other hand,  $N(t)$  with fixed intensity  $\lambda$  generates a dependent claim stream  $X_1, X_2, \dots$  as in (2), where  $\Theta$  is a positive random variable. If the distribution of  $\Theta$  is for instance as in Examples 2.3 and 2.4, then these dependent claims  $X_1, X_2, \dots$  are heavy-tailed. Equivalently, we may view the resulting risk process as

$$R(t) = u + ct - \sum_{k=1}^{N''(t)} Z_k,$$

where  $N''(t)$  is a Poisson process with intensity  $\lambda + \lambda'$  and the marginal density of the claim sizes  $Z_1, Z_2, \dots$  is given by

$$f_Z(x) = \frac{\lambda}{\lambda + \lambda'} \theta e^{-\theta x} + \frac{\lambda'}{\lambda + \lambda'} \nu e^{-\nu x}, \quad x \geq 0. \quad (15)$$

That is, given the realisation of  $\theta$ , the claim size distribution is a mixture of two exponential distributions, but mixed over  $\theta$ , the marginal tail of  $X_1, X_2, \dots$  will determine the tail

behavior if it is heavy-tailed. Since for fixed  $\theta$  the ruin probability  $\psi_\theta(u)$  in the classical risk model with a claim size density of the form (15) has an explicit form as a weighted sum of two exponential terms in  $u$  (see for instance Gerber (1979)), one again obtains an explicit expression for  $\psi(u)$  by virtue of (4). Note that mixing the claim stream  $X_1, X_2, \dots$  with independent (in this case exponential) claims reduces the resulting dependence among the claims. In particular, if Kendall's tau of two arbitrary claims  $X_i, X_j$  is  $\tau_0$ , the value of Kendall's tau of two arbitrary claims  $Z_i, Z_j$  in the new model is  $\tau_1 = (\frac{\lambda}{\lambda + \lambda'})^2 \tau_0$ , because the only way to get positive correlation between two randomly selected claim amounts is to pick two claims coming from the thinned process  $N(t)$ , and because the probability that a claim generated by  $N''(t)$  comes from  $N(t)$  is  $\frac{\lambda}{\lambda + \lambda'}$ . Hence we identified a computational vehicle to determine an explicit ruin probability formula, if the dependence among claim sizes should be weakened, but the marginal tail behavior should stay in the way described by  $N(t)$ , and by choosing the parameter  $\lambda'$  accordingly, one can generate explicit formulas for models with any value of  $\tau_1$  between 0 and  $\tau_0$ . Note that this is done at the expense of losing the explicit Archimedean type dependence structure that was available for  $\lambda' = 0$ . Instead, the joint distribution tail of  $Z_1, \dots, Z_n$  is obtained as follows. Each of the occurred claims  $Z_i$  is of the type  $X_i$  with probability  $\lambda/(\lambda + \lambda')$  and of type  $Y_i$  else. The joint distribution is then the corresponding mixture of the independent  $Y_i$ -types with the Archimedean dependence structure of the  $X_i$ -types.

In complete analogy, one can vary the dependence structure of the inter-occurrence times in the above fashion, still leading to explicit ruin probability formulas.  $\diamond$

**Remark 4.4.** *The resulting risk model in the above example eventually comprised two types of claims: independent light-tailed claims, and dependent heavy-tailed claim amounts (under suitable assumptions on  $\theta$ ). In terms of practical interpretation, this may indeed describe a realistic portfolio situation: the dependence between heavy-tailed claim amounts may come from parameter uncertainty, or from other sources of correlation like environmental risk, climate change and legal risk. In fact, a number of internal models for Solvency II work with regularly varying random losses that are aggregated by a Clayton survival copula.*

**Remark 4.5.** *Inspired by Example 4.3, it is clear that one can generate many more examples of explicit formulas for risk models with dependence by replacing the exponential distribution (given  $\Theta$  in (2)) by a more general distribution for which  $\psi_\theta(u)$  can be determined explicitly (for instance phase-type random variables). The resulting dependence structure and marginal claim size distribution will be a result of the interplay between this choice and the distribution of  $\Theta$  (see Remark 2.6 when to still expect an Archimedean dependence structure).*

## 5. Conclusion and Outlook

In this paper we utilized a simple mixing idea over values of involved parameter to enlarge the class of collective risk models for which explicit representations of the ruin probability are possible. In that way we developed a recipe how to identify explicit solutions for certain combinations of dependence structures and marginal distributions of the involved risks, both for claim sizes and the times of their occurrence. In addition, the method can also be used to derive explicit formulas for dependence between claim

sizes and their inter-occurrence times. It should be noted that the computational vehicle to obtain these explicit formulas need not be the causal reason for the dependence in the model. One may as well find different interpretations of the resulting dependence model and then just use the statistical equivalence to identify the explicit formula. This fact was illustrated in the Examples of Section 2 and 3, where the dependence was introduced through the common factor  $\Theta$  ( $\Lambda$ , respectively), but the resulting dependence structure was an Archimedean copula, which itself may be motivated as an appropriate model by other means. Apart from dependence modeling, the mixing can also be motivated by parameter uncertainty, so that the results can alternatively be interpreted as ruin probabilities when we only have a distribution of an involved parameter available (which for the uncertainty about the Poisson parameter  $\lambda$  was already exploited for credibility-based dynamic premium rules in Bühlmann (1972) and Dubey (1977)).

The approach proposed in this paper can be pushed forward to more general risk models including Markov-additive processes and those Lévy processes for which explicit expressions for the ruin probability exist (see Chapter I.4b of Asmussen and Albrecher (2010) for a list of risk models for which this is the case).

As outlined in the introduction, this versatile recipe can easily be extended beyond the study of ruin probabilities, including Gerber-Shiu functions, the maximum severity of ruin and the expected time-integrated negative part of the risk process (see e.g. Loisel (2005)). Whenever in an independence model an explicit formula for a quantity related to the risk process is available, one can mix over involved parameters and obtain explicit formulas for related models that exhibit a certain degree of dependence among the risks. The resulting skeleton of models for which exact expressions of such quantities are available may help to increase the understanding of the effects of dependence for risk management purposes in general.

**Acknowledgements.** The authors would like to thank Hans-Ulrich Gerber and Dominik Kortschak for helpful remarks. This work has been partially supported by the research chair Actuariat Durable sponsored by Milliman Paris, the French Research National Agency (ANR) under the reference ANR-08-BLAN-0314-01, the Model Management research chair funded by BNP Paribas Assurance, and the Swiss National Science Foundation Project 200021-124635/1.

## References

- [1] H. ALBRECHER AND D. KORTSCHAK, 2009. On ruin probability and aggregate claim representations for Pareto claim size distributions. *Insurance: Mathematics and Economics* 45, No.3, 362–373.
- [2] S. ALINK, M. LOEWE AND M. WUETHRICH, 2004. Diversification of aggregate dependent risks. *Insurance: Mathematics and Economics* 35, No.1, 77–95.
- [3] S. ASMUSSEN AND H. ALBRECHER, 2010. *Ruin Probabilities*. Second Edition, World Scientific, New Jersey.

- [4] H. BÜHLMANN, 1960. Austauschbare stochastische Variablen und ihre Grenzwertsätze. *University of California Publications in Statistics* 3, No.1, 1–36.
- [5] H. BÜHLMANN, 1972. Ruinwahrscheinlichkeit bei erfahrungstarifiziertem Portefeuille. *Bulletin de l'Association des Actuairees Suisses*, No.2, 131–140.
- [6] H. BÜHLMANN AND H. GERBER, 1978. General jump process and time change – or, How to define stochastic operational time. *Scandinavian Actuarial Journal*, No.1, 102–107.
- [7] M. DENUIT, J. DHAENE, M. GOOVAERTS AND R. KAAS, 2005. *Actuarial Theory for Dependent Risks*. Wiley & Sons, Chichester.
- [8] A. DUBEY, 1977. Probabilité de ruine lorsque le paramètre de Poisson est ajusté à posteriori. *Bulletin de l'Association des Actuairees Suisses*, No.2, 211–224.
- [9] W. FELLER, 1970. *An Introduction to Probability and its Applications*. Vol. II. John Wiley & Sons, New York.
- [10] H. GERBER, 1979. *An Introduction to Mathematical Risk Theory*. Huebner Foundation, Illinois.
- [11] J. GRANDPELL, 1992. *Aspects of Risk Theory*. Springer, Heidelberg.
- [12] H. JOE, 1997. *Multivariate Models and Dependence Concepts*. Chapman & Hall, London.
- [13] S.A. KLUGMAN, H.H. PANJER AND G.E. WILLMOT, 2008. *Loss Models: From Data to Decisions*. Third Edition, Wiley & Sons, New Jersey.
- [14] S. LOISEL, 2005. Differentiation of some functionals of risk processes, and optimal surplus allocation. *Journal of Applied Probability* 42, No.2, 379–392.
- [15] R. NELSEN, 1999. *An Introduction to Copulas*. Springer, New York.
- [16] D. OAKES, 1989. Bivariate survival models induced by frailties. *Journal of the American Statistical Association* 84, 487–493.
- [17] C.M. RAMSAY, 2003. A solution to the ruin problem for Pareto distributions. *Insurance: Mathematics and Economics* 33, No.1, 109–116.
- [18] H.C. YEH, 2007. The frailty and the Archimedean structure of the general multivariate Pareto distributions. *Bulletin of the Institute of Mathematics Academia Sinica* 2, No.3, 713–729.